Semi-discrete optimal transport and applications to non-imaging optics

Jocelyn Meyron, Université Grenoble Alpes, LJK
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Motivations: non-imaging optics

**Goal:** design optical components which *transport* light energy
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Goal: design optical components which transport light energy

Applications:
- car beam design (avoid blinding incoming cars)
- luminaire / caustic design (reduce light loss and light pollution)
Motivations: non-imaging optics

**Goal:** design optical components which *transport* light energy

**Applications:**
- car beam design (avoid blinding incoming cars)
- luminaire / caustic design (reduce light loss and light pollution)

We will:
1. Explain the **strong** link between optical component design and optimal transport
2. **Discretize** particular instances of optimal transport to solve these problems
**Introduction: imaging optics**

**Input:** a source $X$, a target $Y$ and a bijection $f : X \rightarrow Y$
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Component (mirror) $S = \text{surface}$
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Component (mirror) $S = $ surface

$T : X \to Y$ models the behaviour of the component when hit by a ray

We can assume $T(x) = F(x, \vec{n}_S(x))$ and $F = $ Snell’s law
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We can assume $T(x) = F(x, \vec{n}_S(x))$ and $F = \text{Snell’s law}$

**Problem:** Find $S$ such that $F(x, \vec{n}_S(x)) = f(x)$ for all $x \in X$
Non-imaging optics: the bijection $f$ is not an input anymore
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- One approach: estimate $f$ with a heuristic and integrate the normals $\vec{n}_S$

Possible idea: use **optimal transport** to determine $f$ but still a heuristic

[Schwartzburg '14, Feng, Froese, Liang '16]
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 target normals

[Schwartzburg '14, Feng, Froese, Liang '16]
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  $\Rightarrow$ Goal: prescribe areas of facets $\approx$ reflected intensity in a direction
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- One approach: estimate $f$ with a heuristic and integrate the normals $n_S$
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- Example of a discretization of a non-imaging optics problem:

  $$\langle x, \langle x | p(y) \rangle - \psi(y) \rangle$$

  \[ \implies \text{Goal: prescribe areas of facets } \approx \text{reflected intensity in a direction} \]

  \[ \implies \text{Observation: } \psi = \text{dual variable} \text{ in an *optimal transport* problem} \]

  \[ \implies \text{dual variable gives a } \text{parametrization} \text{ of the mirror } S \]
Introduction: non-imaging optics

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- One approach: estimate \( f \) with a heuristic and integrate the normals \( \vec{n}_S \)
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- Example of a discretization of a non-imaging optics problem:

- We focus on semi-discrete optimal transport:
  - Efficient numerical methods
  - Regularity of the solutions: convexity \( \implies \) important for the fabrication
Overview

A. Optimal transport
   1. Generalities on optimal transport
   2. Semi-discrete optimal transport
   3. OT between a triangulation and a point cloud

B. Optimal transport and non-imaging optics
   1. Light Energy Conservation equation
   2. Generic and parameter-free algorithm
   3. Numerical results
**Goal:** Find a mass-preserving mapping $T : X \rightarrow Y$ between two probability measures $\mu$ and $\nu$ minimizing a transport cost $c$
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Monge formulation: minimize $\int_X c(x, T(x)) \, d\mu(x)$

where $T$ is a transport map between $\mu$ and $\nu$ $(M)$
Optimal transport: applications

OT provides a means to measure distances between measures

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▶ Interpolation between surfaces

[Lévy et al ’17]
Optimal transport: applications

OT provides a means to measure \textbf{distances} between measures

\textbf{Applications:}

- Interpolation between surfaces
- Inverse problems: reconstruction of the early universe, shape matching...

[Brenier et al '03 (pictures by B. Lévy)]

[Feydy et al '17]
Optimal transport: applications

OT provides a means to measure distances between measures

Applications:

- Interpolation between surfaces
- Inverse problems: reconstruction of the early universe, shape matching...
- Partial differential equations: fluid mechanics...

[de Goes et al '15]
Kantorovich relaxation

\[ \text{minimize } \int_X c(x, T(x)) \, d\mu(x) : T \text{ transport map between } \mu \text{ and } \nu \quad (M) \]

- Monge formulation: *no solutions* even for simple problems and *non-linear*
  \[\Rightarrow\text{ idea: replace the transport map } T \text{ by a probability measure } \gamma \text{ on } X \times Y\]
  \[\Rightarrow \text{ transport plan } \gamma(A \times B) = \text{ amount of mass moved from } A \text{ to } B\]
Kantorovich relaxation

minimize $\int_X c(x, T(x)) \, d\mu(x) : T$ transport map between $\mu$ and $\nu$ \hspace{1cm} (M)

- Monge formulation: *no solutions* even for simple problems and non-linear
  ⟷ idea: replace the transport map $T$ by a probability measure $\gamma$ on $X \times Y$
  ⟷ **transport plan** $\gamma(A \times B) =$ amount of mass moved from $A$ to $B$

Kantorovich formulation: minimize $\int_{X \times Y} c(x, y) \, d\gamma(x, y)$
where $\gamma \in \text{Prob}(X \times Y)$ such that $(P_X)^\# \gamma = \mu$ and $(P_Y)^\# \gamma = \nu$ \hspace{1cm} (K)

⟹ *linear* programming problem with convex constraints ⟷ existence of solutions
Kantorovich relaxation

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\textit{Kantorovich} formulation: minimize \[\int_{X \times Y} c(x, y) \, d\gamma(x, y)\]
where \(\gamma \in \text{Prob}(X \times Y)\) such that \((P_X)_#\gamma = \mu\) and \((P_Y)_#\gamma = \nu \quad (K)\)

\[\implies \text{linear programming problem with convex constraints } \implies \text{existence of solutions}\]

\textit{Dual problem}: maximize \[\int_X \phi(x) \, d\mu(x) - \int_Y \psi(y) \, d\nu(y)\]
where \(\phi \in C^0(X), \psi \in C^0(Y)\) and \(\phi(x) - \psi(y) \leq c(x, y) \quad (K^*)\)
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where \(\phi \in C^0(X), \psi \in C^0(Y)\) and \(\phi(x) - \psi(y) \leq c(x, y)\) \quad (K^*)

- We introduce: \(\psi^c(x) = \inf_{y \in Y} [c(x, y) + \psi(y)]\) to remove the constraint

\[
\text{maximize } \int_X \psi^c(x) \, d\mu(x) - \int_Y \psi(y) \, d\nu(y) \text{ where } \psi \in C^0(Y) \quad (K^{**})
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Setting

Input: $\mu$ probability measure on $X$ and $\nu = \sum_{i=1}^{N} \nu_i \delta_{y_i}$ on $Y = \{y_1, \ldots, y_N\}$

Finding optimal transport between $\mu$ and $\nu$:

$$\max \Phi(\psi) := \int_X \inf_{1 \leq i \leq N} (c(x, y_i) + \psi_i) \, d\mu(x) - \sum_{i=1}^{N} \nu_i \psi_i \quad (K^{**})$$

$\Phi$ is called the **Kantorovich functional**
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$\Phi$ is called the **Kantorovich functional**

**Definition:** Laguerre cell of $y_i$: $\text{Lag}_i(\psi) = \{x \in X \mid \forall j, c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j\}$
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**Definition:** For $\psi \in \mathbb{R}^N$, we define $T_\psi : x \in X \mapsto \arg\min_{1 \leq i \leq N} (c(x, y_i) + \psi_i) \in Y$
**Discrete Monge-Ampère equation**

Recall: \( \Phi(\psi) = \sum_{i=1}^{N} \int_{\text{Lag}_i(\psi)} (c(x, y_i) + \psi_i) \, d\mu(x) - \sum_{i=1}^{N} \nu_i \psi_i \)

**Theorem:** Regularity of \( \Phi \)

If \( \mu \) is AC and verifies the (Neg) condition, then \( \Phi \) is concave and \( C^1 \) and

\[
\frac{\partial \Phi}{\partial \psi_i}(\psi) = G_i(\psi) - \nu_i \quad \text{where} \quad G_i(\psi) := \mu(\text{Lag}_i(\psi))
\]

**Corollary:** \( T_\psi \) is an optimal transport map between \( \mu \) and \( \nu \)

\[\iff \quad \psi \text{ is a maximizer of } \Phi \]
\[\iff \quad \nabla \Phi(\psi) = 0 \]
\[\iff \quad \forall i \in \{1, \ldots, N\}, \ G_i(\psi) = \nu_i \quad (\text{DMA})\]
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Numerical methods?
Damped Newton Algorithm: description

Recall: $G : \psi \in \mathbb{R}^N \mapsto (\mu(\text{Lag}_i(\psi)))_{1 \leq i \leq N} \in \mathbb{R}^N$

Admissible domain: $E_\varepsilon := \{\psi \in \mathbb{R}^N \mid \forall i, G_i(\psi) \geq \varepsilon\}$

[Mirebeau '15]
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Damped Newton algorithm for solving (DMA)

Input: $\psi^0 \in \mathbb{R}^N$ s.t. $\varepsilon := \frac{1}{2} \min_{1 \leq i \leq N} \min(G_i(\psi^0), \nu_i) > 0$

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Loop: Compute Newton direction: $v^k := -DG(\psi^k) + (G(\psi^k) - \nu)$

Choose $\ell$ so that $\psi^{k+1} := \psi^k + 2^{-\ell} v_k \in E_\varepsilon$

and $\|G(\psi^{k+1}) - \nu\| \leq (1 - 2^{(\ell+1)}) \|G(\psi^k) - \nu\|$

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- Damping
  - and $\|G(\psi^{k+1}) - \nu\| \leq (1 - 2^{(\ell+1)})\|G(\psi^k) - \nu\|$

$\implies$ **Convergence when** $X$ **is a triangulated surface?**

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**Damping**

$\rho(\text{Lag}_i(\psi)) \geq \varepsilon$
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OT between a triangulation and a point cloud

Input:
- A prob. measure on a triangulation $X$ in $\mathbb{R}^d$, $\mu = \sum_{\sigma} \mu_\sigma$, where $\sigma = \text{triangle}$
- A prob. measure on a point cloud $Y \subset \mathbb{R}^d$, $\nu = \sum_{i=1}^{N} \nu_i \delta_{y_i}$
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Output:
- Transport plan between $\mu$ and $\nu$ for quadratic cost $\rightsquigarrow$ Laguerre cells $(\text{Lag}_i(\psi))_{1 \leq i \leq N}$
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Question: can we use still use the Newton method to solve OT between $\mu$ and $\nu$?
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- Transport plan between \( \mu \) and \( \nu \) for quadratic cost \( \leadsto \) Laguerre cells \( (\text{Lag}_{i}(\psi))_{1 \leq i \leq N} \)

Question: can we use still use the Newton method to solve OT between \( \mu \) and \( \nu \)?

- \( \mu \) not AC anymore \( \implies \) Brenier's theorem does not apply anymore!
  \( \implies \) Optimal transport may not be unique or even exists
OT between a triangulation and a point cloud

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- A prob. measure on a triangulation $X$ in $\mathbb{R}^d$, $\mu = \sum_{\sigma} \mu_{\sigma}$, where $\sigma = \text{triangle}$
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Output:
- Transport plan between $\mu$ and $\nu$ for quadratic cost $\leadsto$ Laguerre cells $(\text{Lag}_i(\psi))_{1 \leq i \leq N}$

Question: can we use still use the Newton method to solve OT between $\mu$ and $\nu$?

- $\mu$ not AC anymore $\implies$ Brenier's theorem does not apply anymore!
- $\implies$ Optimal transport may not be unique or even exists

Solution: use a genericity assumption on the point cloud $Y$ and regularity on $\mu$
Main theorem

**Theorem:**
Assume $\mu$ is a regular simplicial measure

$y_1, \cdots, y_N$ are in generic position

Then the damped Newton method converges with linear rate globally i.e.

$$\|G(\psi^k) - \nu\| \leq (1 - \tau^* k )\|G(\psi^0) - \nu\| \quad \text{where} \quad \tau^* \in ]0, 1]$$

[Mérigot, M., Thibert, SIIMS '18]
Main theorem

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Assume $\mu$ is a regular simplicial measure and $y_1, \cdots, y_N$ are in generic position.
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\|G(\psi^k) - \nu\| \leq (1 - \frac{\tau^*}{2})^k \|G(\psi^0) - \nu\| \text{ where } \tau^* \in ]0, 1]\] 

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$$\|G(\psi^k) - \nu\| \leq (1 - \frac{\tau^*}{2})^k \|G(\psi^0) - \nu\|$$

where $\tau^* \in ]0, 1]$ [Mérigot, M., Thibert, SIIMS '18]

**Genericity position:** example of non-generic case, edge of $\sigma \perp (y_i y_j)$

$$\frac{\partial G_i}{\partial \psi_j}(\psi^1) \propto \mu(\partial \text{Lag}_i(\psi^1) \cap \sigma) > 0$$
Main theorem

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Assume $\mu$ is a regular simplicial measure and $y_1, \cdots, y_N$ are in generic position.

Then the damped Newton method converges with linear rate globally i.e.

$$\|G(\psi^k) - \nu\| \leq (1 - \frac{\tau^*}{2})^k \|G(\psi^0) - \nu\| \text{ where } \tau^* \in ]0, 1]$$

**Genericity position:** example of non-generic case, edge of $\sigma \perp (y_i y_j)$

$$\text{Lag}_i(\psi^2) \quad \text{Lag}_j(\psi^2)$$

$$\frac{\partial G_i}{\partial \psi_j}(\psi^2) \propto \mu(\partial \text{Lag}_i(\psi^2) \cap \sigma) = 0$$

$$\implies G \text{ not } C^1$$
Numerical results

Optimal transport for a uniform source

Initial: $\psi^0$

Final
Numerical results

Optimal transport for a uniform source

Initial: $\psi^0$

Final
Numerical results

- Optimal quantization of a probability measure on a triangulated surface

- Remeshing with respect to a density $\mu$ (uniform, mean curvature)
Numerical results

- Optimal quantization of a probability measure on a triangulated surface

- Remeshing with respect to a density $\mu$ (uniform, mean curvature)
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Four non-imaging optics problems

Mirror $\mathcal{R}$

Collimated source

$\mathbb{R}^2 \times \{0\}$

Target light $S^2$
Four non-imaging optics problems

Mirror $R$

Collimated source

$R^2 \times \{0\}$

Target light

$S^2$

Lens $R$

Collimated source

$R^2 \times \{0\}$

$S^2$

Target light
Four non-imaging optics problems

- **Mirror $\mathcal{R}$**
  - Collimated source
  - Target light

- **Lens $\mathcal{R}$**
  - Collimated source
  - Target light

- **Point source**
  - Target light

- **Target light**
  - $\mathbb{S}^2$
  - $\mathbb{R}^2 \times \{0\}$
Four non-imaging optics problems

- **Collimated source**
  - Mirror $\mathcal{R}$
  - Target light
  - $\mathbb{R}^2 \times \{0\}$
  - $\mathbb{S}^2$

- **Point source**
  - Mirror $\mathcal{R}$
  - Target light
  - 0
  - $\mathbb{S}^2$

- **Collimated source**
  - Lens $\mathcal{R}$
  - Target light
  - $\mathbb{R}^2 \times \{0\}$
  - $\mathbb{S}^2$

- **Point source**
  - Lens $\mathcal{R}$
  - Target light
  - 0
  - $\mathbb{S}^2$
Four non-imaging optics problems

- **Collimated source**
  - **Mirror $\mathcal{R}$**
  - **Target light**

- **Point source**
  - **Mirror $\mathcal{R}$**
  - **Target light**

- **Collimated source**
  - **Lens $\mathcal{R}$**

- **Point source**
  - **Lens $\mathcal{R}$**

*(concave too)*
Mirror design for a collimated source

Input: collimated light source $\mu$ and target light at infinity $\nu = \sum_i \nu_i \delta y_i$

Goal: design a convex mirror $\mathcal{R}$ that sends $\mu$ to $\nu$
Mirror design for a collimated source

**Input:** collimated light source $\mu$ and target light at infinity $\nu = \sum_i \nu_i \delta_{y_i}$

**Goal:** design a convex mirror $\mathcal{R}$ that sends $\mu$ to $\nu$

1. **Discretization:** $\mu$ supported on a triangulation $X$ and $\nu$ on a point cloud $Y$, where $y_i \in S^2$
Mirror design for a collimated source

**Input:** collimated light source $\mu$ and target light at infinity $\nu = \sum_i \nu_i \delta_{y_i}$

**Goal:** design a convex mirror $R$ that sends $\mu$ to $\nu$

2. $R$ is convex and can be parametrized by $R_\psi(x) = (x, \max_{1\leq i\leq N} \langle x | p_i \rangle - \psi_i)$
Mirror design for a collimated source

**Input:** collimated light source $\mu$ and target light at infinity $\nu = \sum_i \nu_i \delta_{y_i}$

**Goal:** design a convex mirror $\mathcal{R}$ that sends $\mu$ to $\nu$

3. $V_i(\psi) = \{x \in X \mid x \text{ reflected towards } y_i\}$ and $G_i(\psi) = \mu(V_i(\psi))$

$\mathcal{R}$

$X$

$y_i \in S^2$

known slope  unknown elevation
Mirror design for a collimated source

**Input:** collimated light source $\mu$ and target light at infinity $\nu = \sum_i \nu_i \delta_{y_i}$

**Goal:** design a convex mirror $\mathcal{R}$ that sends $\mu$ to $\nu$

3. $V_i(\psi) = \{x \in X \mid x \text{ reflected towards } y_i\}$ and $G_i(\psi) = \mu(V_i(\psi))$

Find $\psi \in \mathbb{R}^N$ such that $\forall i \in \{1, \ldots, N\}, \ G_i(\psi) = \nu_i$ (LEC)
Mirror design for a collimated source

**Input:** collimated light source $\mu$ and target light at infinity $\nu = \sum_i \nu_i \delta_{y_i}$

**Goal:** design a convex mirror $R$ that sends $\mu$ to $\nu$

4. $V_i(\psi) = \{x \in X \mid \forall j, -\langle x|p_i \rangle + \psi_i \leq -\langle x|p_j \rangle + \psi_j \} = \text{Lag}_i(\psi)$ for $c(x, p) = -\langle x|p \rangle$

Find $\psi \in \mathbb{R}^N$ such that $\forall i \in \{1, \ldots, N\}, G_i(\psi) = \nu_i$ (LEC)
Goal: solve (LEC) where

- $V_i(\psi)$ is a *Laguerre* cell $\implies$ optimal transport problem for some cost $c$
- $X \subset \mathbb{R}^2 \times \{0\}$ for collimated lights and $X \subset \mathbb{S}^2$ for point lights
- $\mathcal{R}_{\psi}$ is a parametrization of the component
  - *piecewise affine* function for mirror & lens / collimated light
  - *pieces of paraboloids* for mirror / point light
  - *pieces of ellipsoids* for lens / point light
Overview

A. Optimal transport
   1. Generalities on optimal transport
   2. Semi-discrete optimal transport
   3. OT between a triangulation and a point cloud

B. Optimal transport and non-imaging optics
   1. Light Energy Conservation equation
   2. Generic and parameter-free algorithm
   3. Numerical results
Generic algorithm

**Algorithm:** Mirror / lens construction

**Input**
- A light source $X, \mu$
- A target light $Y, \nu$
- A tolerance $\eta > 0$
- A parametrization function $\psi \mapsto \mathcal{R}_\psi$
- A transformation function $\tau : \text{Lag} \mapsto X \cap \text{Pow}$

**Output**
- A triangulation $\mathcal{R}_T$ of a mirror or lens $\mathcal{R}$

*depends on the optical design problem*
Algorithm: Mirror / lens construction

Input
- A light source $X, \mu$
- A target light $Y, \nu$
- A tolerance $\eta > 0$
- A parametrization function $\psi \mapsto \mathcal{R}_\psi$
- A transformation function $\tau : \text{Lag} \mapsto X \cap \text{Pow}$

Output
- A triangulation $\mathcal{R}_T$ of a mirror or lens $\mathcal{R}$

Step 1: Initialization

$\psi^0 \leftarrow \text{INITIAL}_\text{WEIGHTS}(Y)$ i.e. $\psi^0 \in E_\varepsilon$

depends on the optical design problem
**Generic algorithm**

**Algorithm:** Mirror / lens construction

**Input**
- A light source $X, \mu$
- A target light $Y, \nu$
- A tolerance $\eta > 0$
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- A transformation function $\tau : \text{Lag} \mapsto X \cap \text{Pow}$

**Output**
- A triangulation $\mathcal{R}_T$ of a mirror or lens $\mathcal{R}$

**Step 1** Initialization

$\psi^0 \leftarrow \text{INITIAL_WEIGHTS}(Y)$ i.e. $\psi^0 \in E_\varepsilon$

**Step 2** Solve $G(\psi) = \sigma$

$\psi \leftarrow \text{DAMPED_NEWTON}(X, \mu, Y, \nu, \psi^0, \eta, \tau)$

depends on the optical design problem
Generic algorithm

Algorithm: Mirror / lens construction

Input
- A light source $X, \mu$
- A target light $Y, \nu$
- A tolerance $\eta > 0$
- A parametrization function $\psi \mapsto R_\psi$
- A transformation function $\tau : \text{Lag} \mapsto X \cap \text{Pow}$

Output
- A triangulation $R_T$ of a mirror or lens $R$

Step 1 Initialization
$\psi^0 \leftarrow \text{INITIAL_WEIGHTS}(Y)$ i.e. $\psi^0 \in E_\varepsilon$

Step 2 Solve $G(\psi) = \sigma$
$\psi \leftarrow \text{DAMPED_NEWTON}(X, \mu, Y, \nu, \psi^0, \eta, \tau)$

Step 3 Construct a triangulation $R_T$ of $R$
$R_T \leftarrow \text{SURFACE_CONSTRUCTION}(\psi, R_\psi)$
Overview

A. Optimal transport
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Numerical results: mirror design / collimated light

Target image: $N = 256 \times 256$ Diracs
Numerical results: mirror design / collimated light

Target image: $N = 256 \times 256$ Diracs

Mirror $\mathcal{R}$

Collimated source

$\mathbb{R}^2 \times \{0\}$

Target light $S^2$

Laguerre diagram and mesh

Reflected image
Numerical results: other settings

Collimated source

Target light

Lens $\mathcal{R}$

$\mathbb{R}^2 \times \{0\}$

Mirror $\mathcal{R}$

Point source

Target light

Lens $\mathcal{R}$

$\mathbb{S}^2$

Point source

$\mathbb{S}^2$

Target light

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Numerical results: physical prototypes
Summary & Perspectives

- **Optimal transport** can be used to **unify** non-imaging optics problems.
- **Optimal transport** can be solved **very efficiently** using Newton algorithm.
Summary & Perspectives

- **Optimal transport** can be used to **unify** non-imaging optics problems
- **Optimal transport** can be solved **very efficiently** using Newton algorithm

Thank you for your attention