Semi-discrete optimal transport and applications to non-imaging optics

Jocelyn Meyron, Université Grenoble Alpes, LJK

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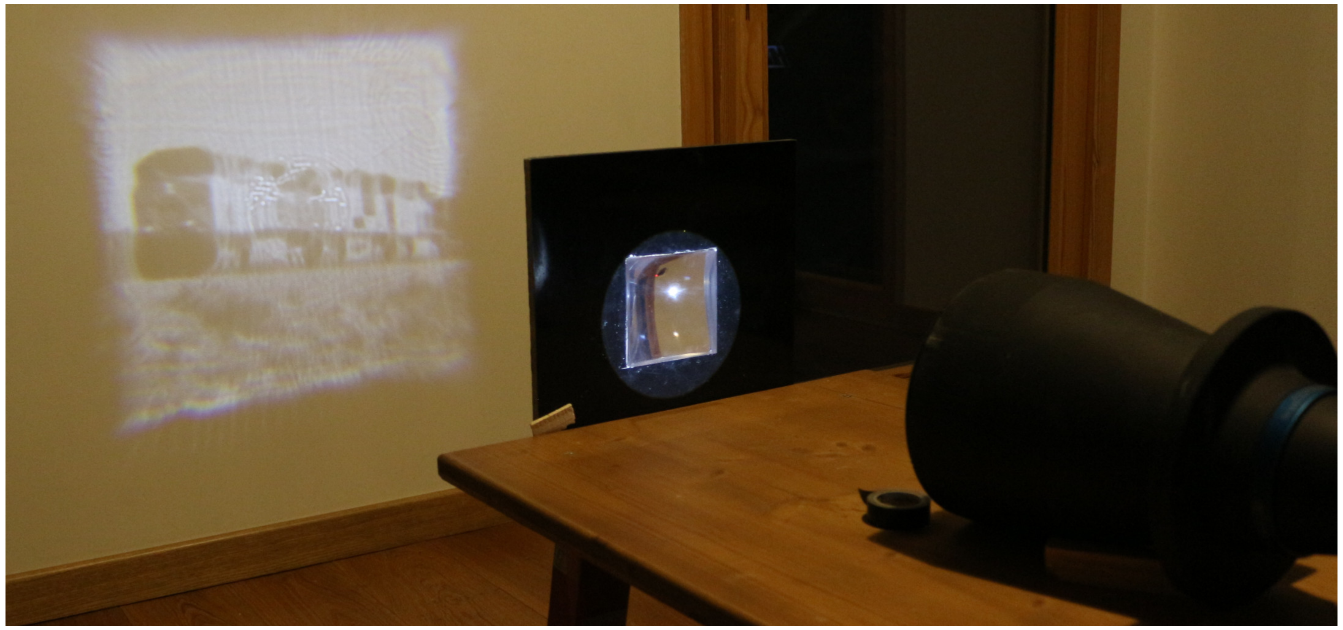
Motivations: non-imaging optics

Goal: design optical components which *transport* light energy



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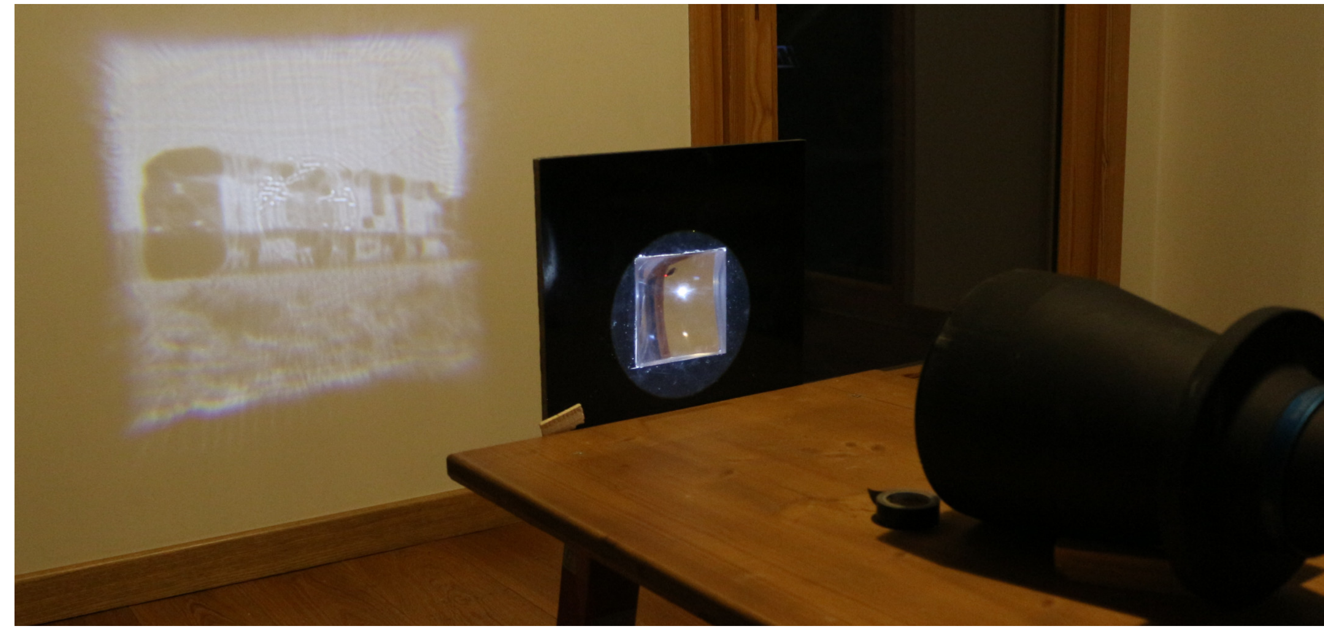

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- luminaire / caustic design (reduce light loss and light pollution)

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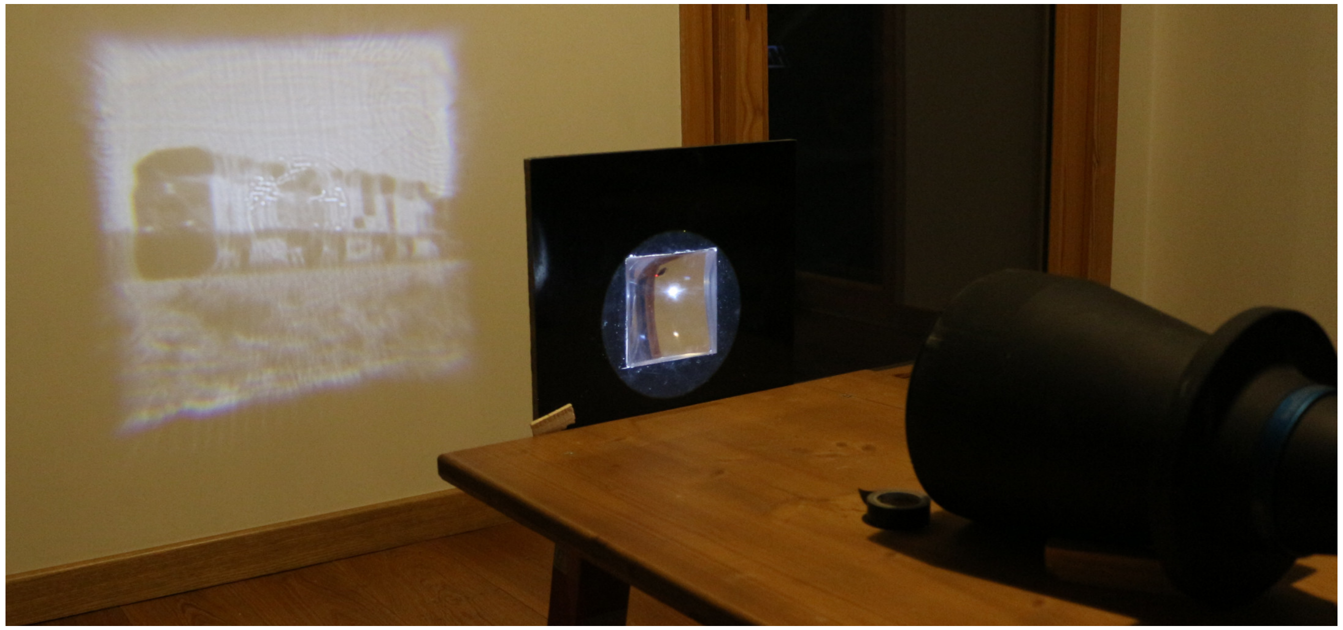

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We will:

1. Explain the **strong** link between optical component design and optimal transport
2. **Discretize** particular instances of optimal transport to solve these problems

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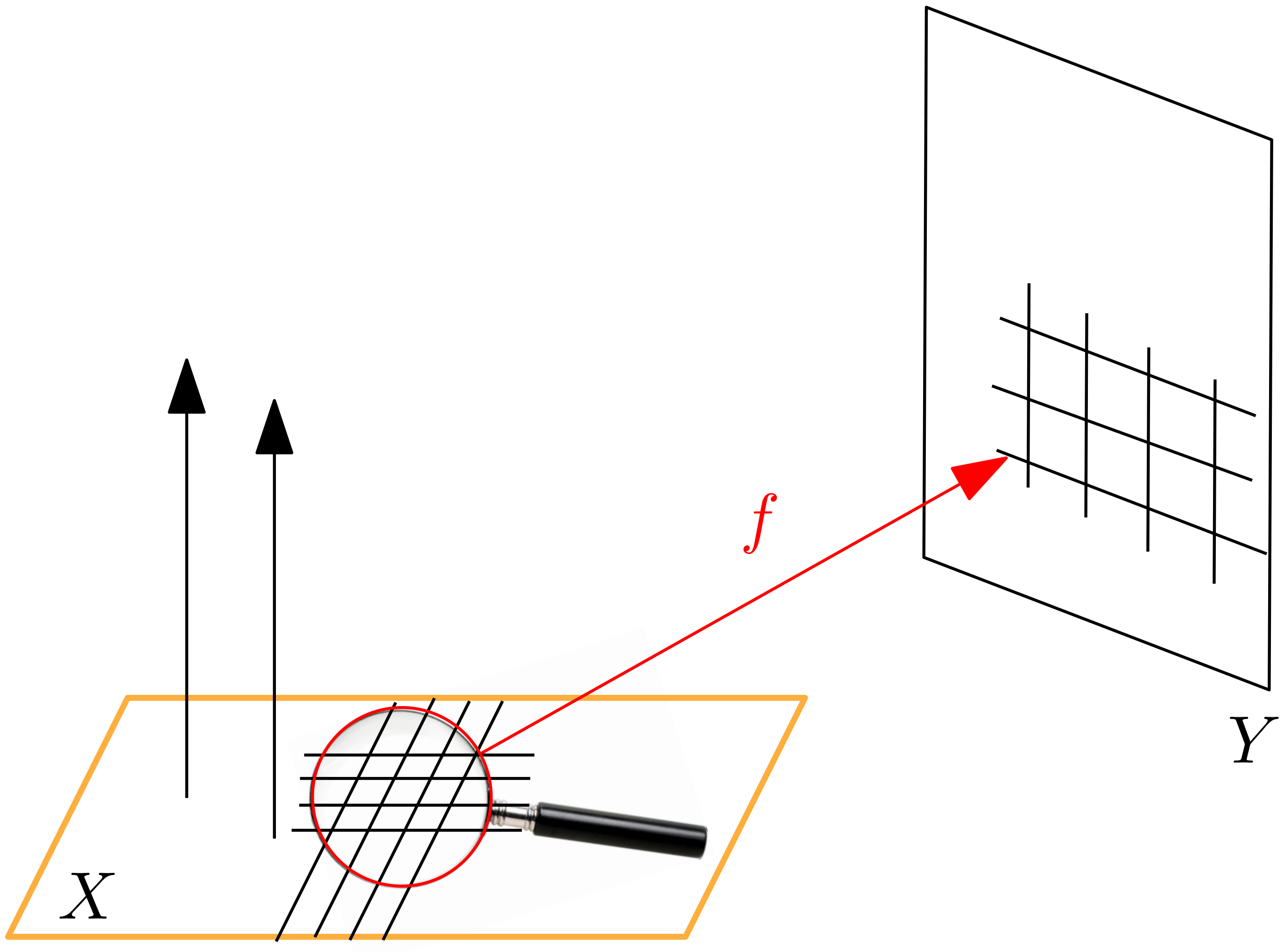
Input: a source X, a target Y and a bijection $f : X \to Y$



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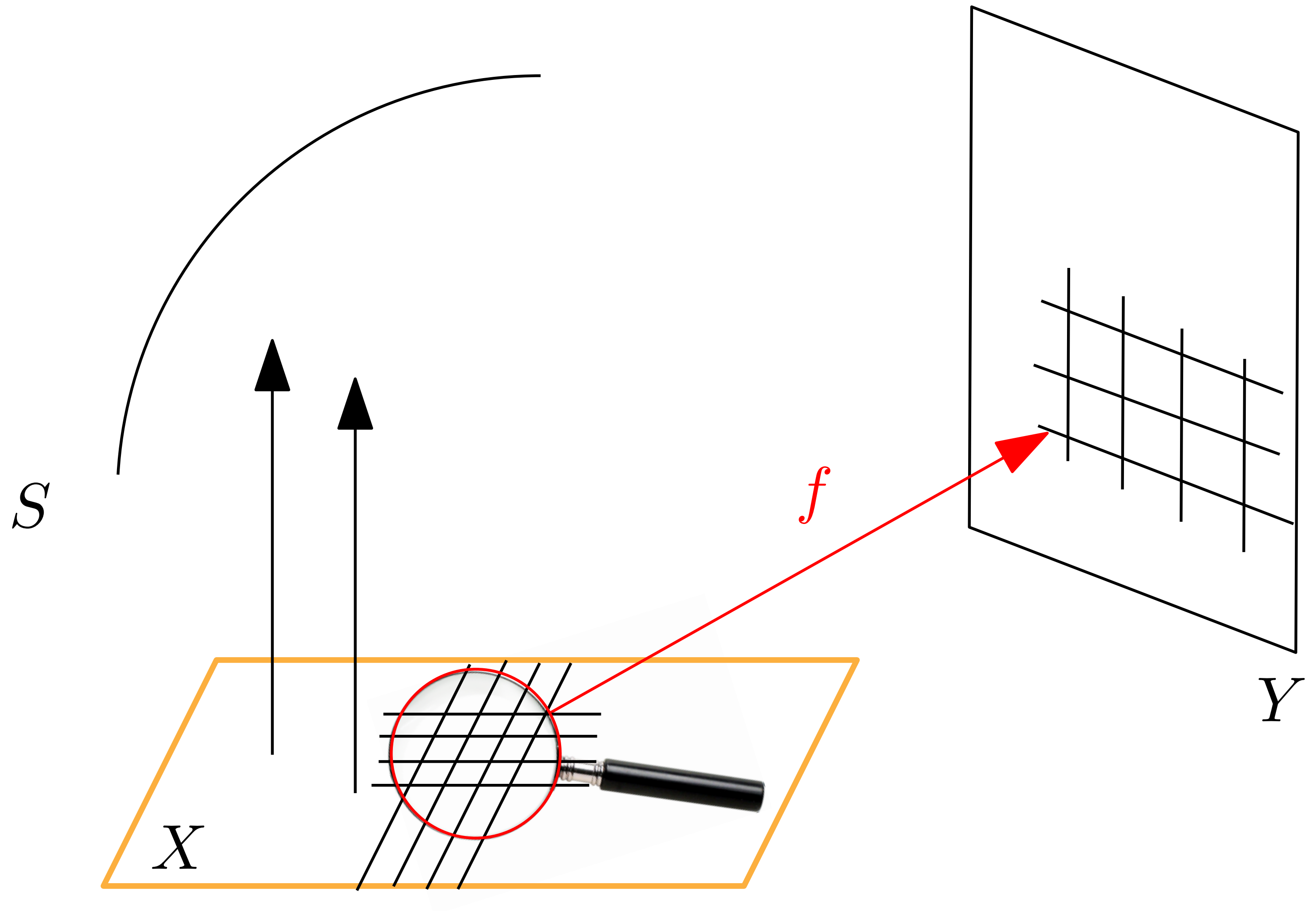

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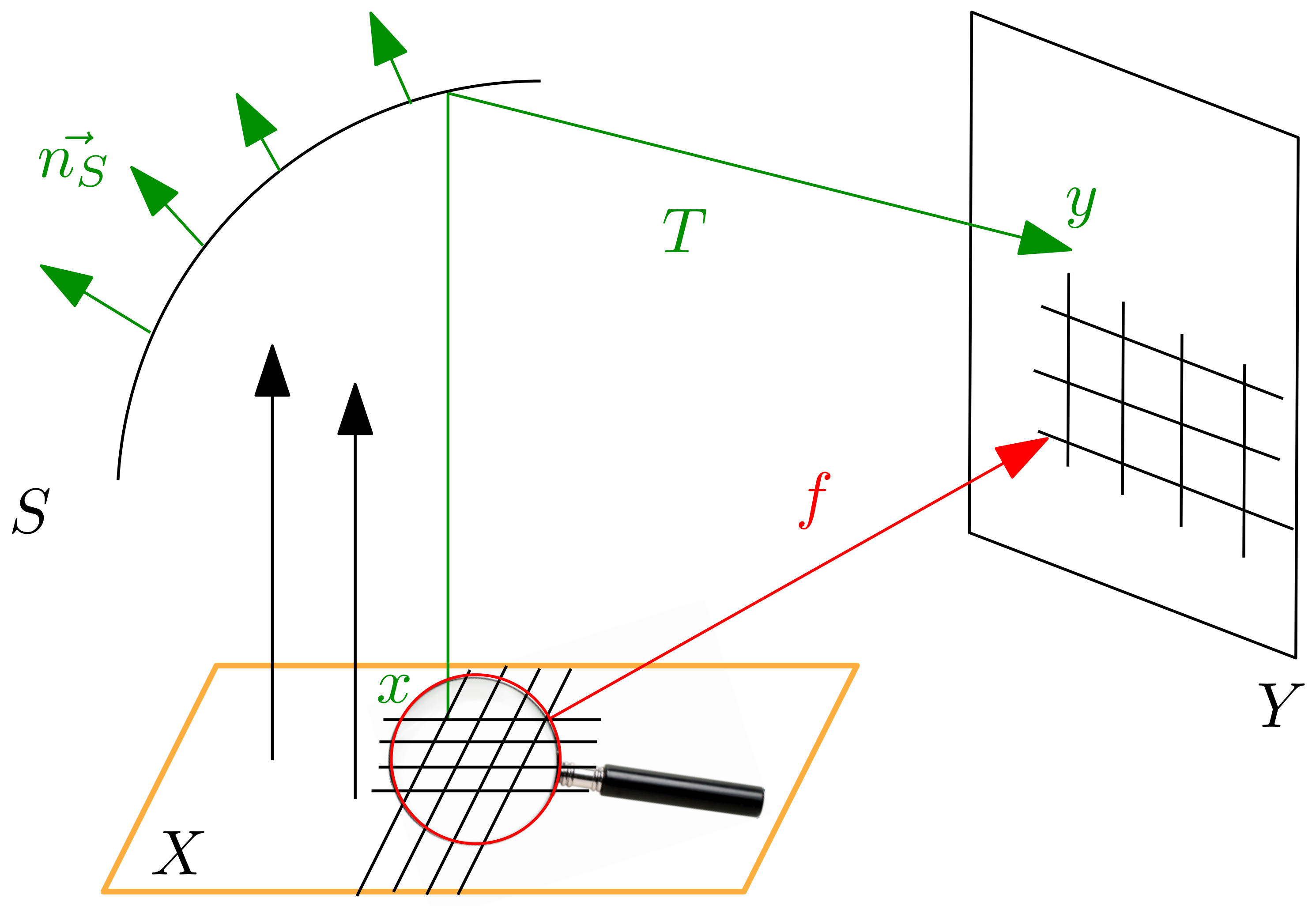
We can assume $T(x) = F(x, \vec{n_S}(x))$ and F = Snell's law



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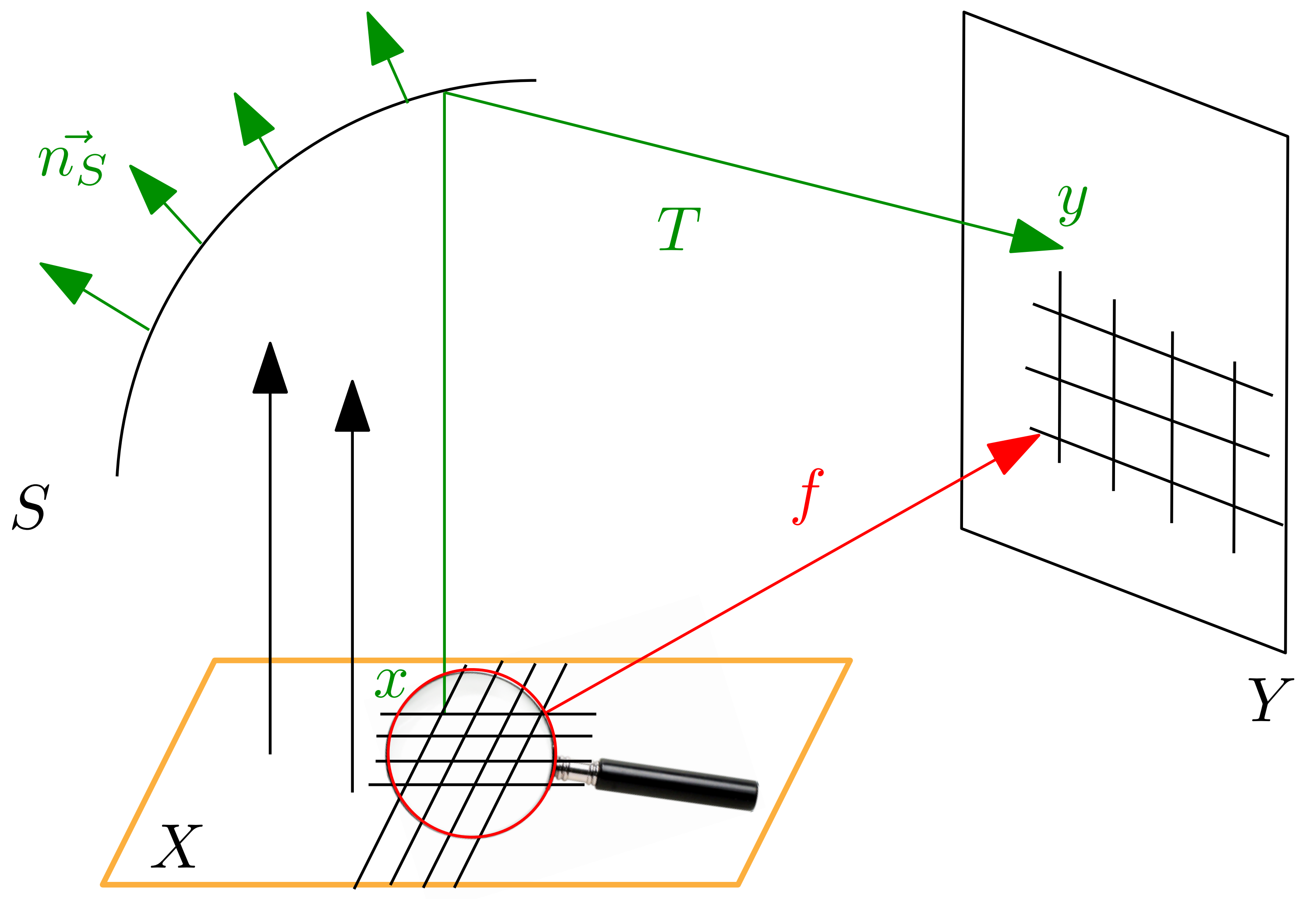
Problem: Find S such that $F(x, \vec{n_S}(x)) = f(x)$ for all $x \in X$



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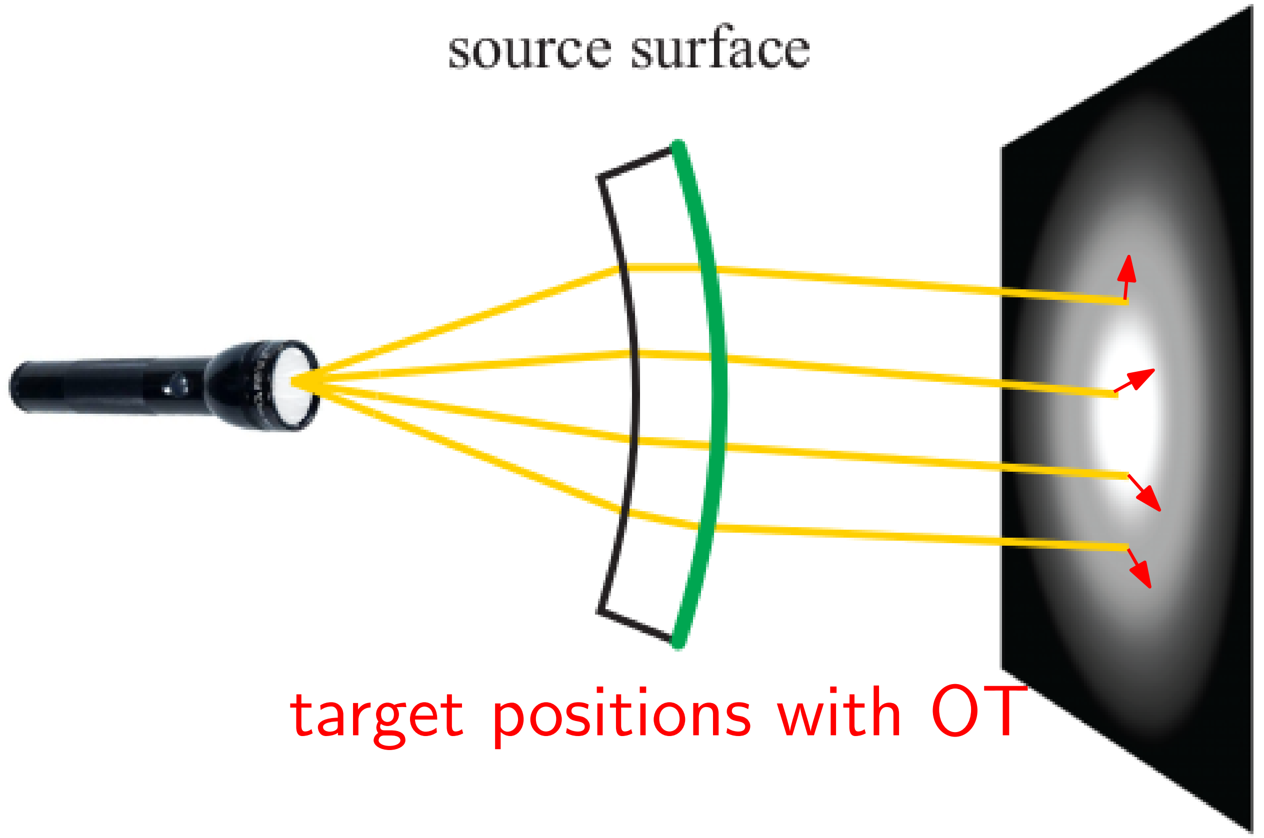

[Schwartzburg '14, Feng, Froese, Liang '16]

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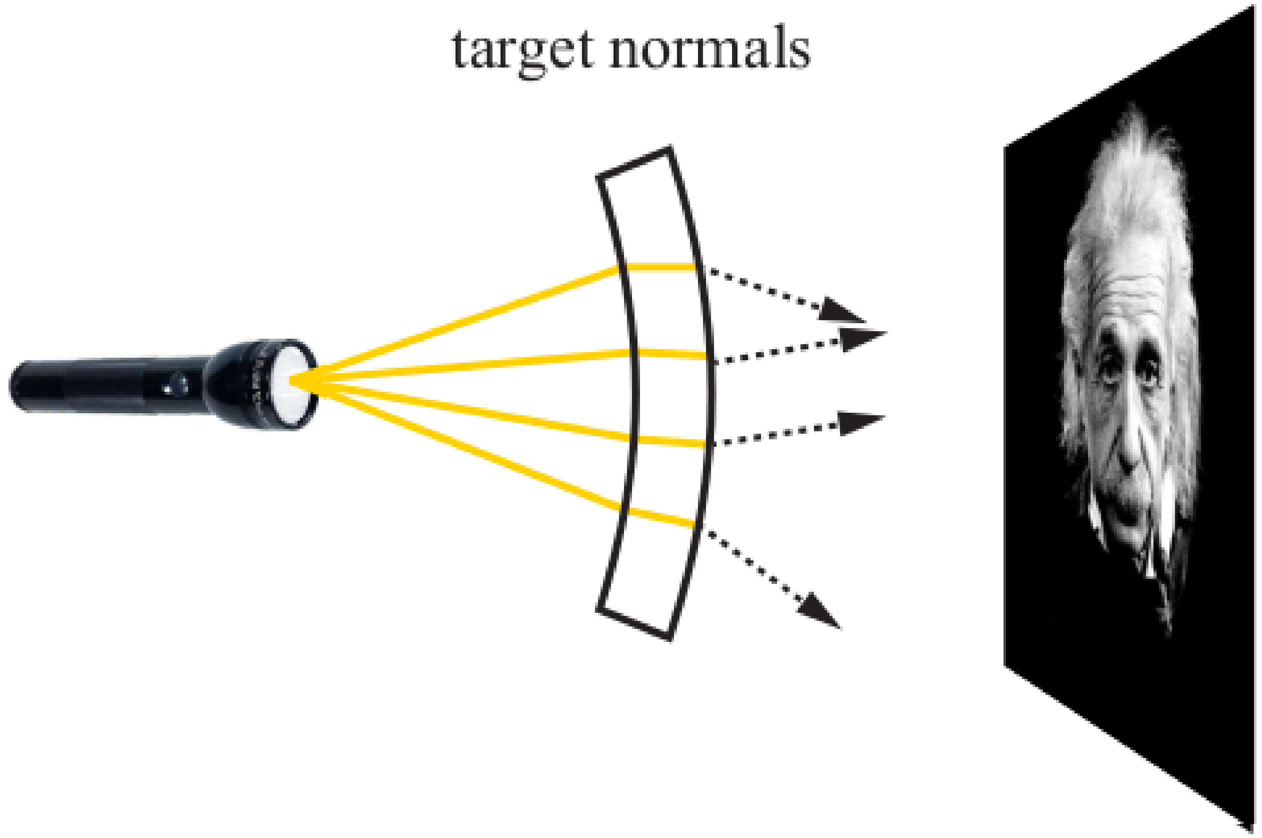

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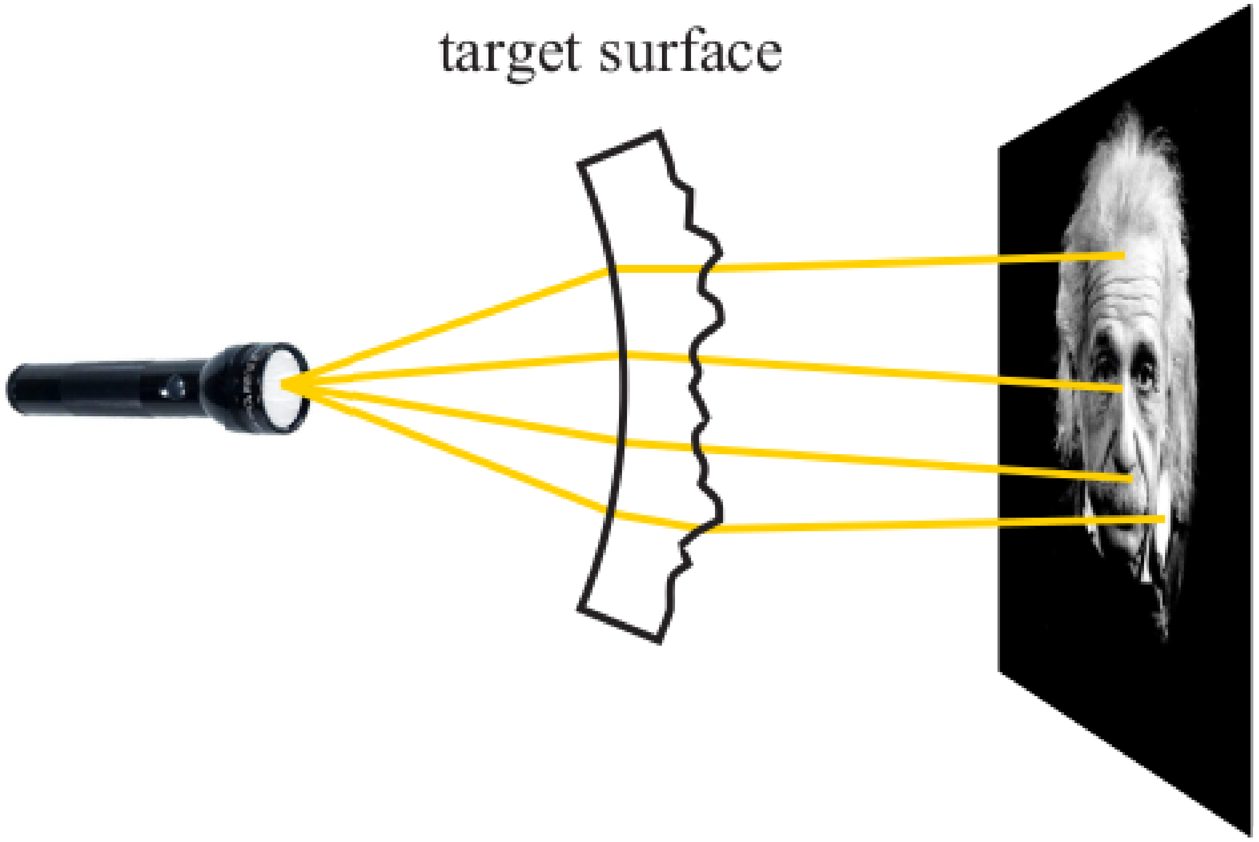

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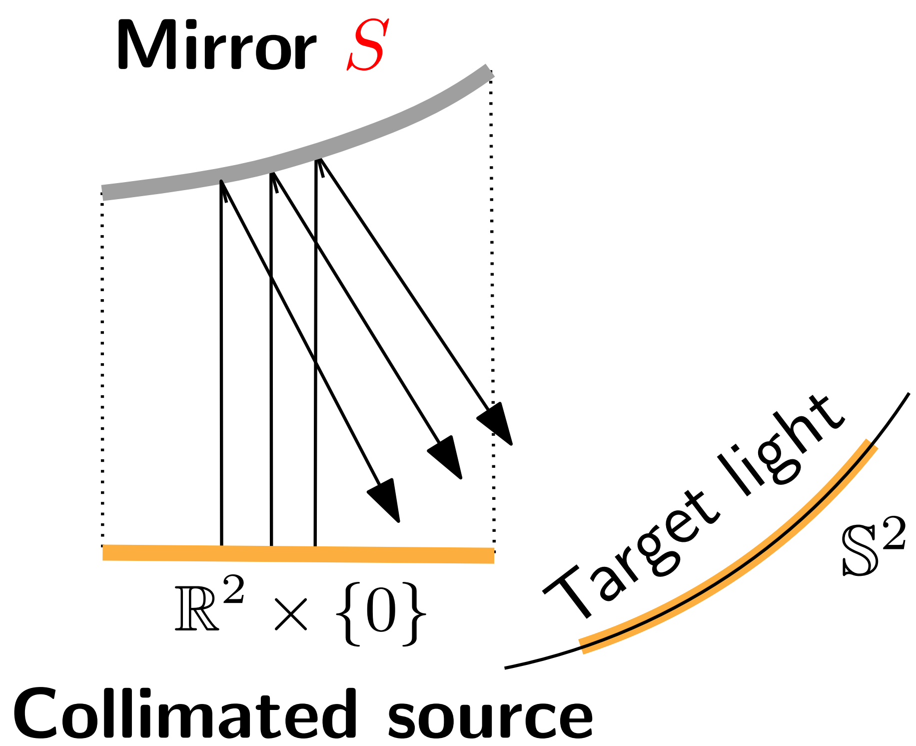

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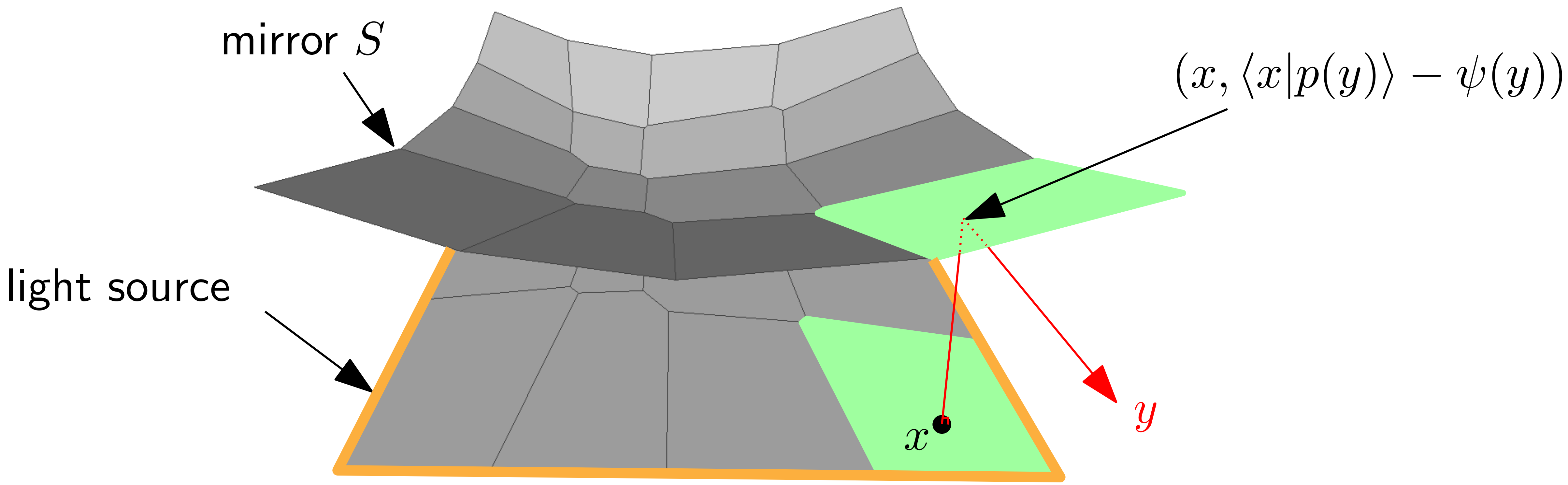

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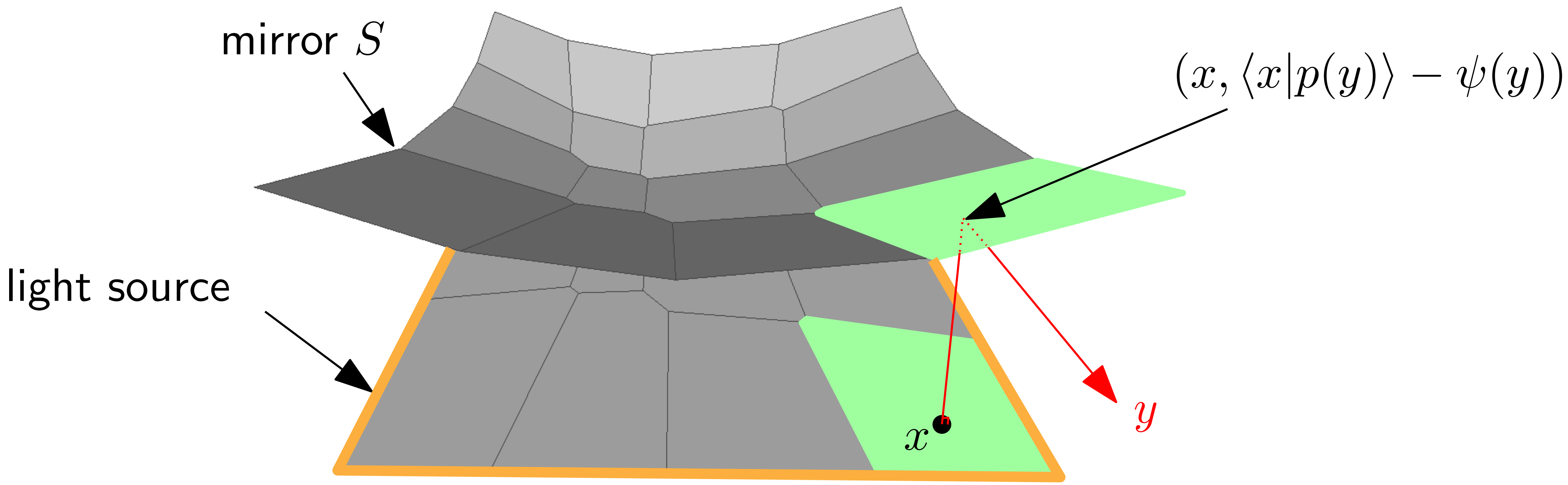

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Observation: ψ = **dual variable** in an *optimal transport* problem

$\Longrightarrow$ dual variable gives a **parametrization** of the mirror S

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- Example of a discretization of a non-imaging optics problem:

- We focus on **semi-discrete** optimal transport:
  - Efficient numerical methods
  - Regularity of the solutions: **convexity** $\implies$ important for the fabrication

Overview

A. Optimal transport

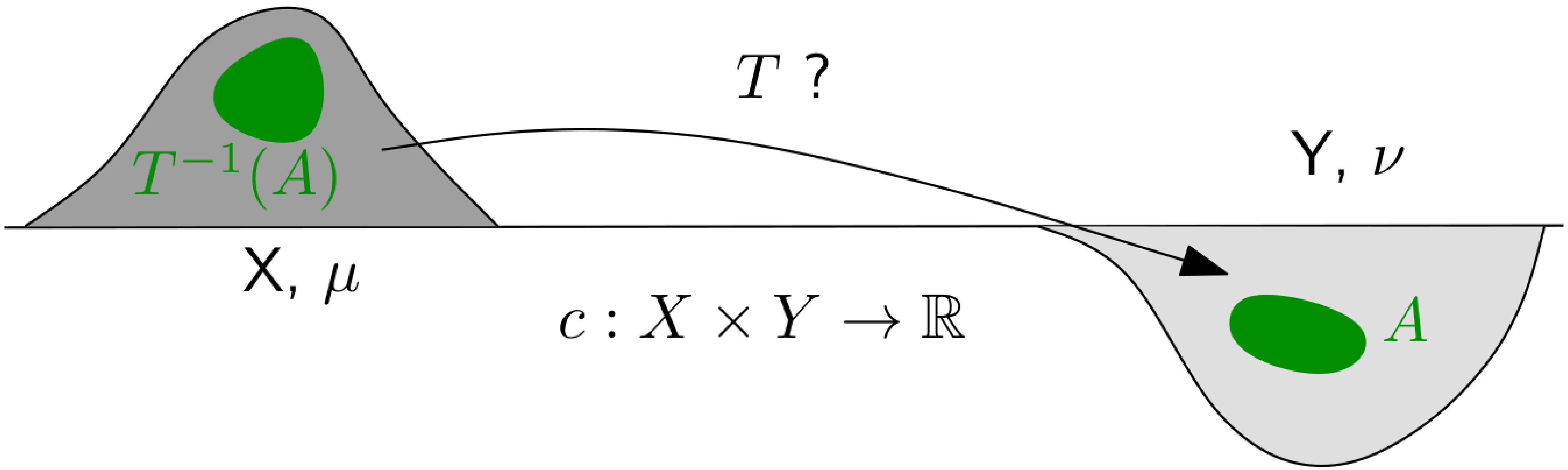
1. Generalities on optimal transport
2. Semi-discrete optimal transport
3. OT between a triangulation and a point cloud

B. Optimal transport and non-imaging optics

1. Light Energy Conservation equation
2. Generic and parameter-free algorithm
3. Numerical results

Optimal transport: introduction

Goal: Find a mass-preserving mapping $T : X \to Y$ between two probability measures μ and ν minimizing a transport cost c



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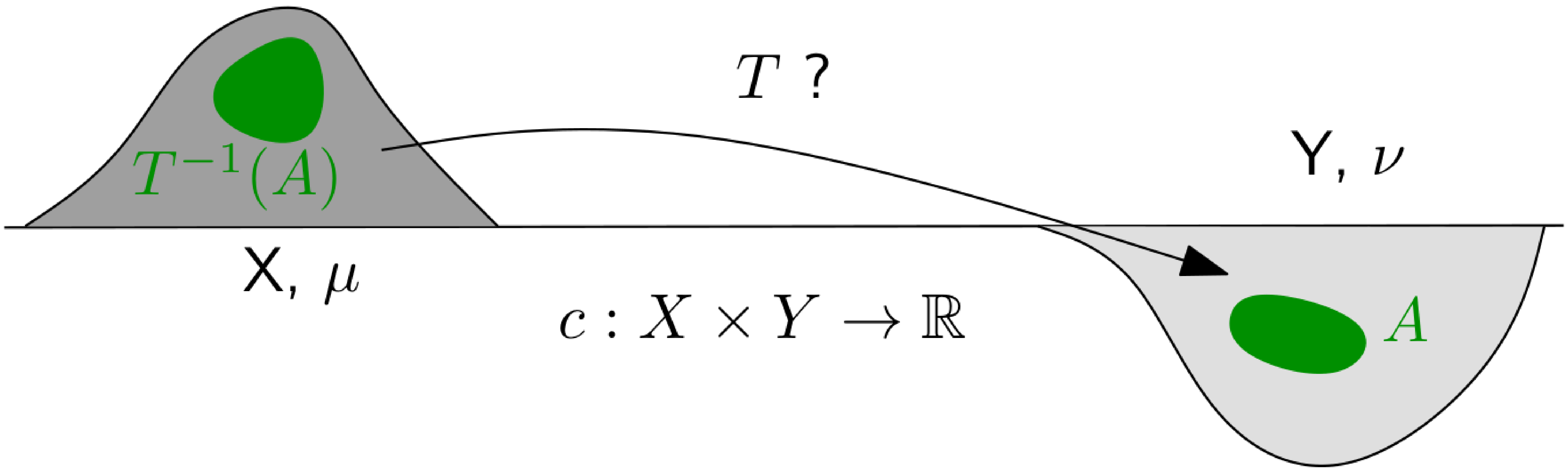

Definition: T is a **transport map** between μ and ν if

$$T_{\#}\mu = \nu \text{ meaning } \forall A \subset Y,\ \mu(T^{-1}(A)) = \nu(A)$$

Monge formulation: minimize $\int_X c(x, T(x)) \, \mathrm{d}\, \mu(x)$
where T is a transport map between μ and ν $\quad (M)$

Optimal transport: applications

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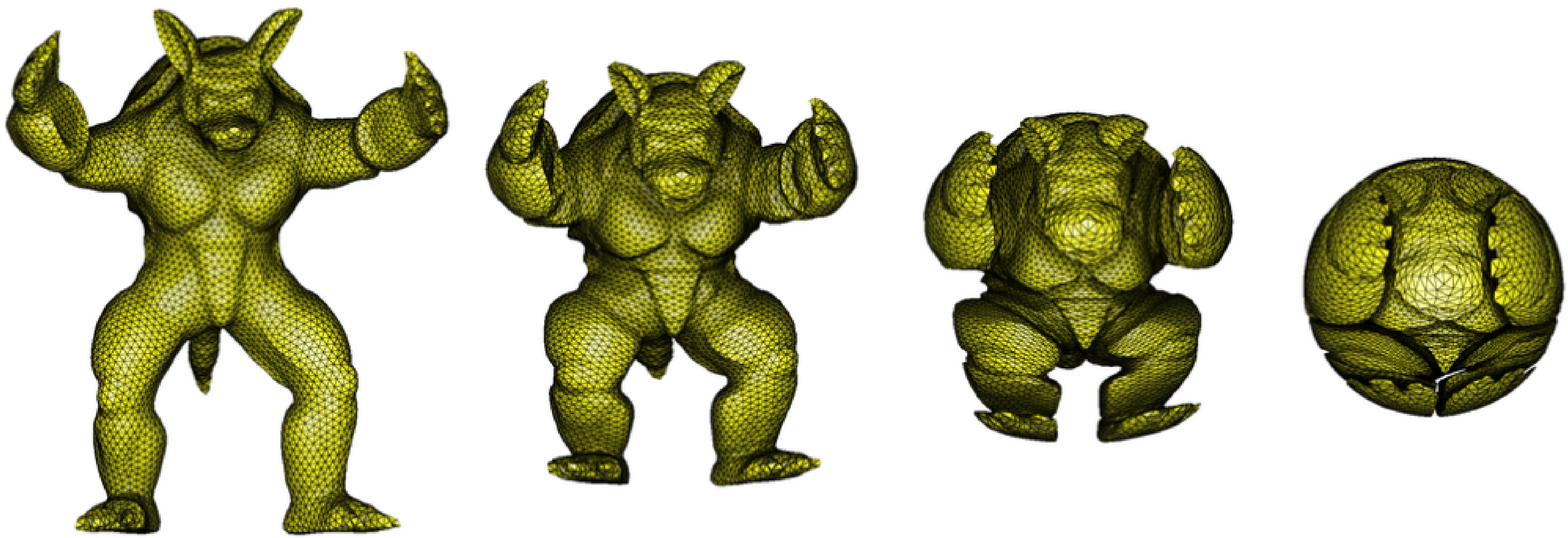
- Interpolation between surfaces



[Lévy et al '17]

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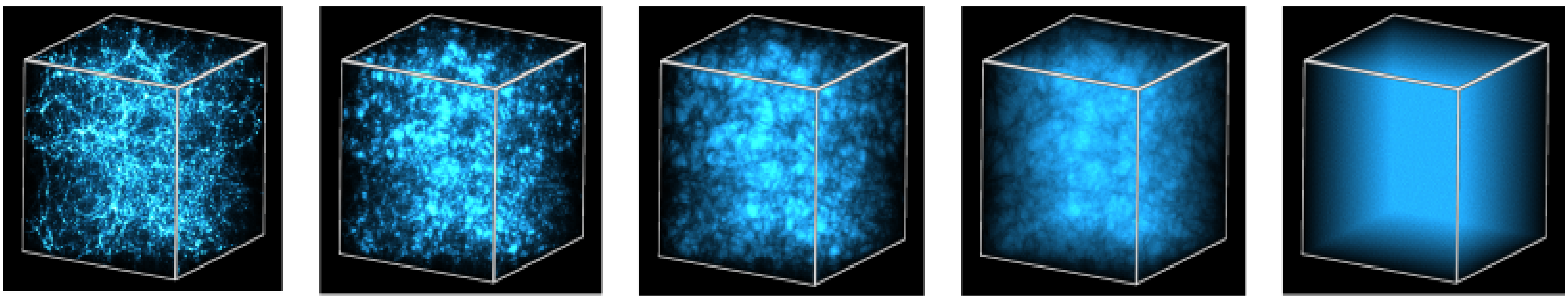
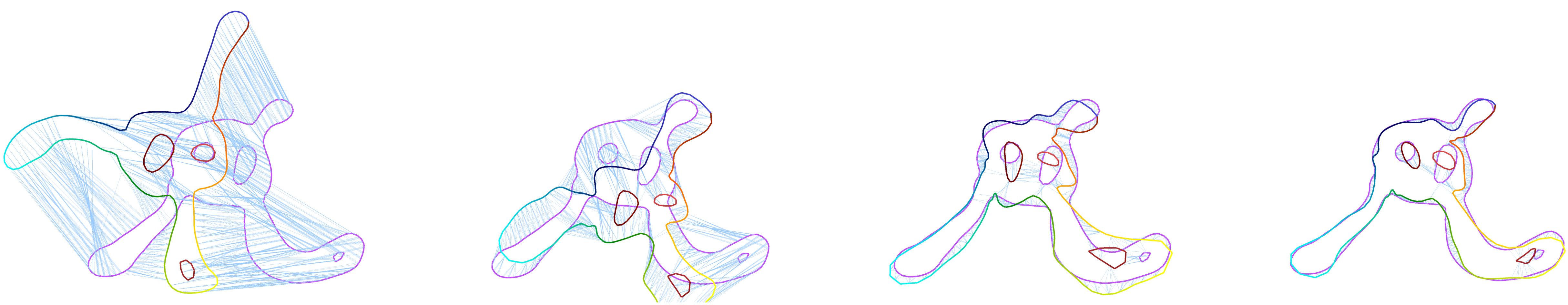


[Brenier et al '03 (pictures by B. Lévy)]



[Feydy et al '17]

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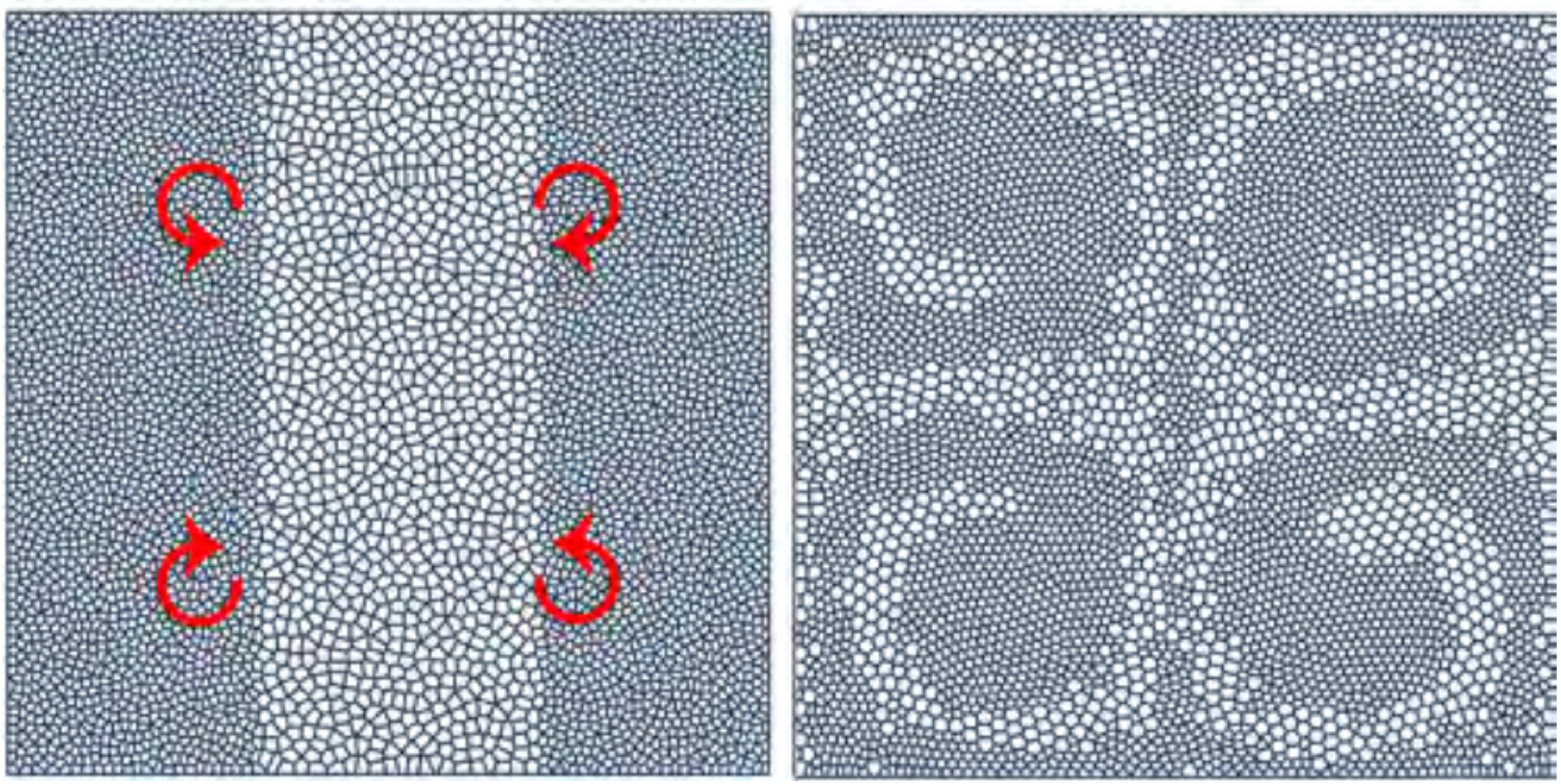
- Interpolation between surfaces
- Inverse problems: reconstruction of the early universe, shape matching...
- Partial differential equations: fluid mechanics...



[de Goes et al '15]

Kantorovich relaxation

minimize $\int_X c(x, T(x)) \, \mathrm{d}\, \mu(x)$: T transport map between μ and ν $\quad (M)$

- Monge formulation: *no solutions* even for simple problems and *non-linear*
  $\implies$ idea: replace the transport map T by a probability measure γ on $X \times Y$
  $\implies$ **transport plan** $\gamma(A \times B) =$ amount of mass moved from A to B

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Kantorovich formulation: minimize $\int_{X\times Y} c(x,y)\,\mathrm{d}\,\gamma(x,y)$
where $\gamma \in \mathrm{Prob}(X \times Y)$ such that $(P_X)_{\#}\gamma = \mu$ and $(P_Y)_{\#}\gamma = \nu$ (K)

$\Longrightarrow$ *linear* programming problem with convex constraints $\Longrightarrow$ existence of solutions

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Dual problem: maximize $\int_X \phi(x)\,\mathrm{d}\,\mu(x) - \int_Y \psi(y)\,\mathrm{d}\,\nu(y)$
where $\phi \in \mathcal{C}^0(X)$, $\psi \in \mathcal{C}^0(Y)$ and $\phi(x) - \psi(y) \le c(x, y)$ (K^*)

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- We introduce: $\psi^c(x) = \inf_{y \in Y} [c(x, y) + \psi(y)]$ to remove the constraint

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Input: μ probability measure on X and $\nu = \sum_{i=1}^{N} \nu_i \delta_{y_i}$ on $Y = \{y_1, \ldots, y_N\}$

Finding optimal transport between μ and ν:

maximize $\Phi(\psi) := \int_X \inf_{1 \leq i \leq N} (c(x, y_i) + \psi_i) \, \mathrm{d}\, \mu(x) - \sum_{i=1}^{N} \nu_i \psi_i$ $\quad (K^{**})$

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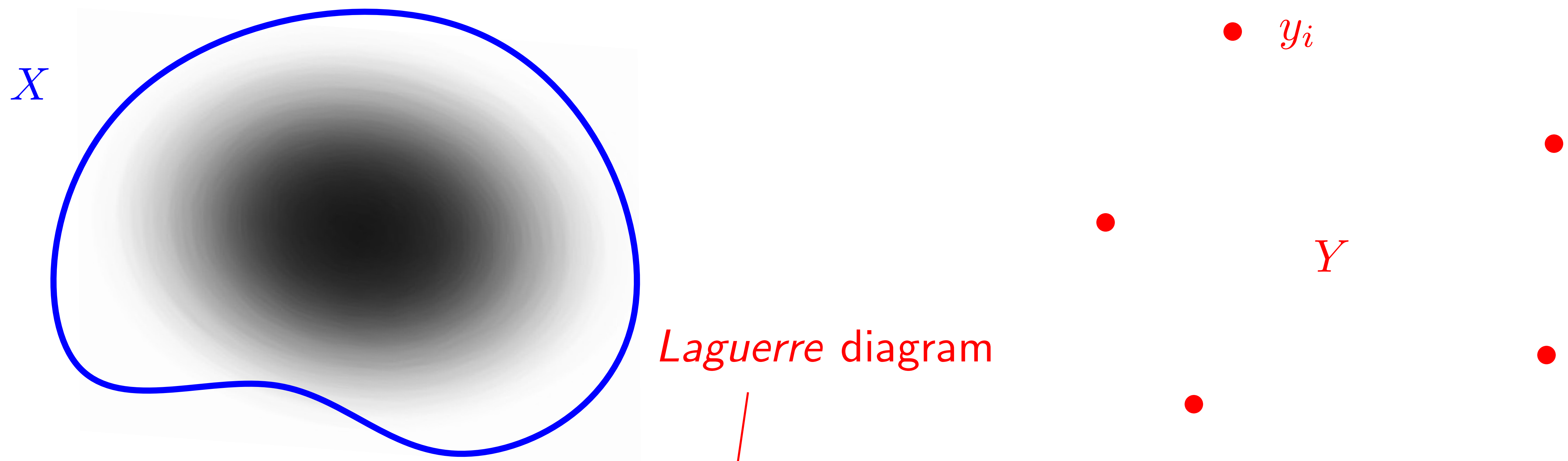

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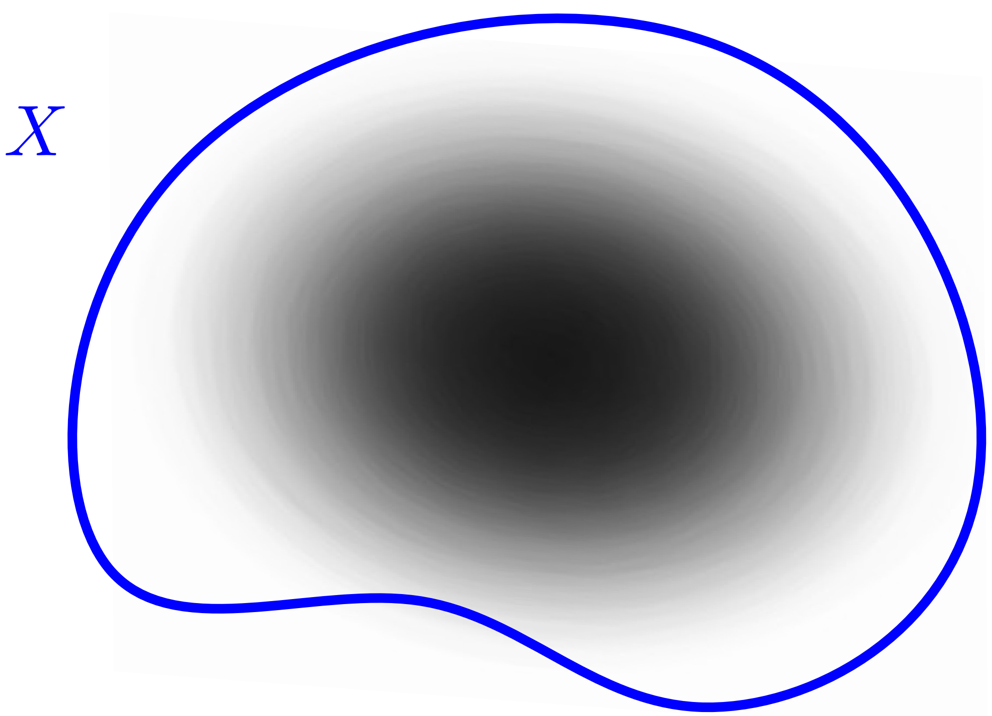

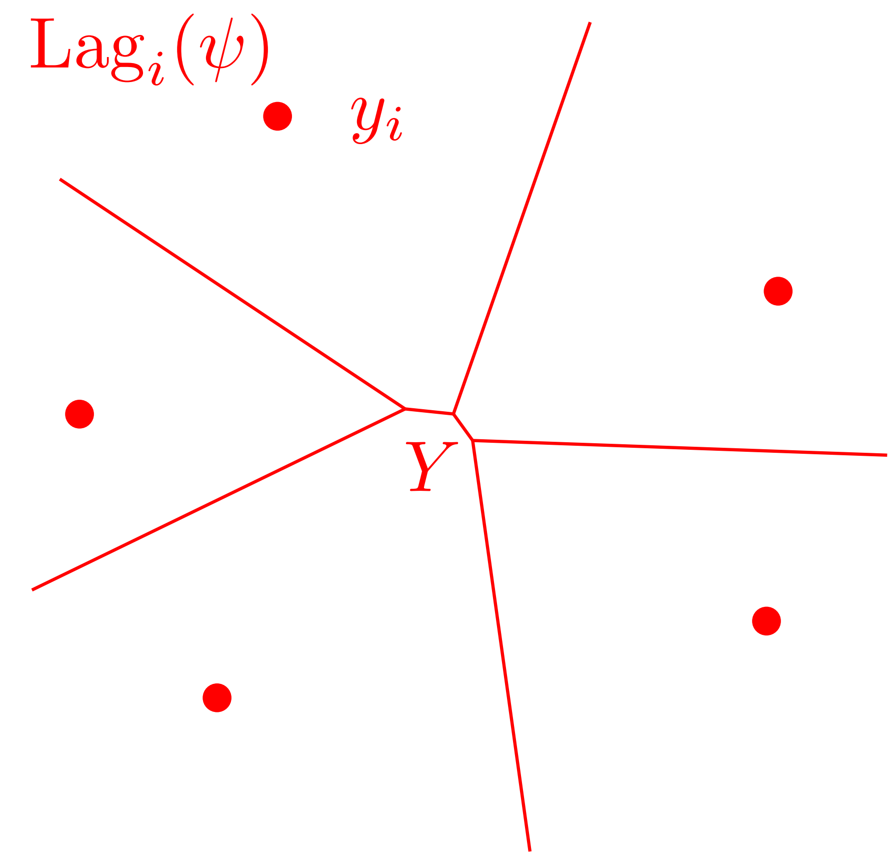

Laguerre diagram

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Definition: Laguerre cell of y_i: $\mathrm{Lag}_i(\psi) = \{x \in X \mid \forall j,\ c(x, y_i) + \psi_i \le c(x, y_j) + \psi_j\}$

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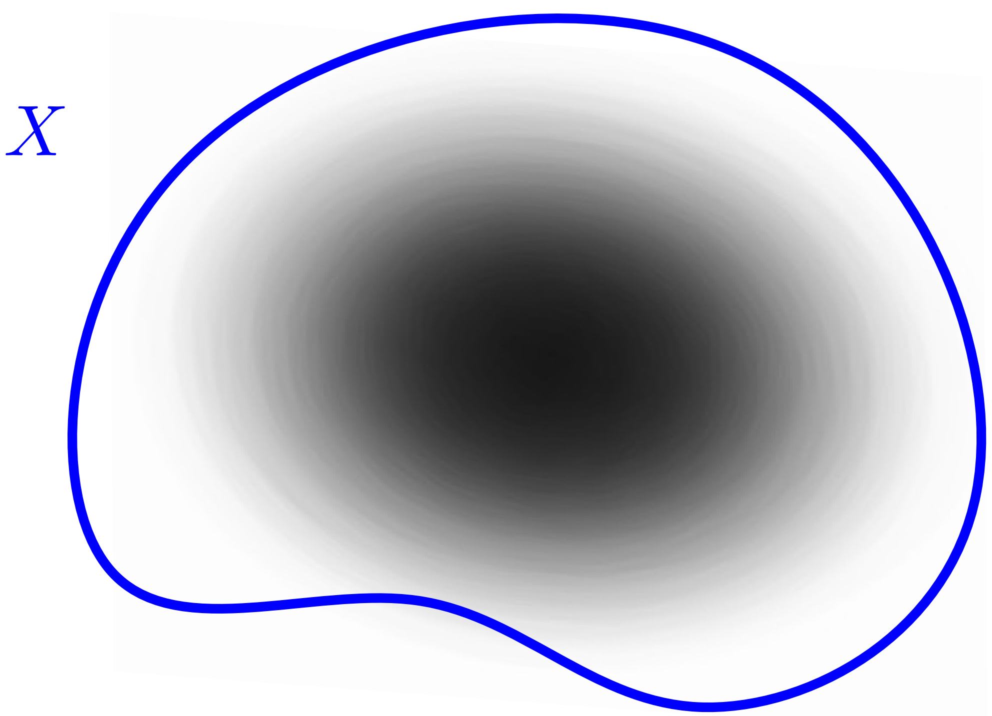

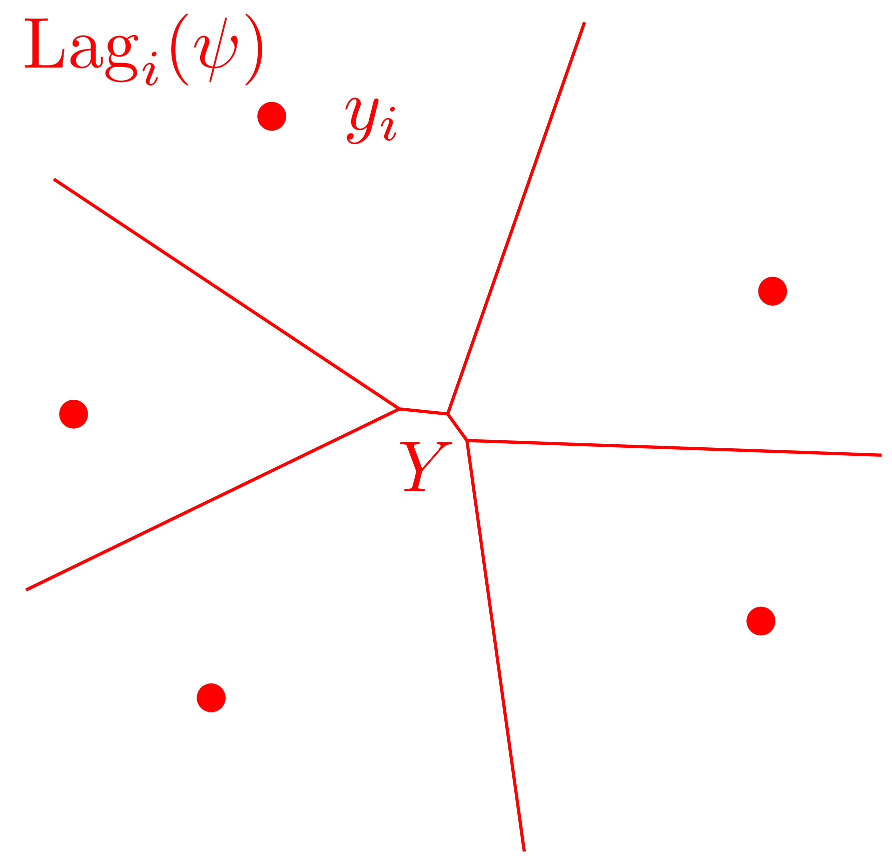

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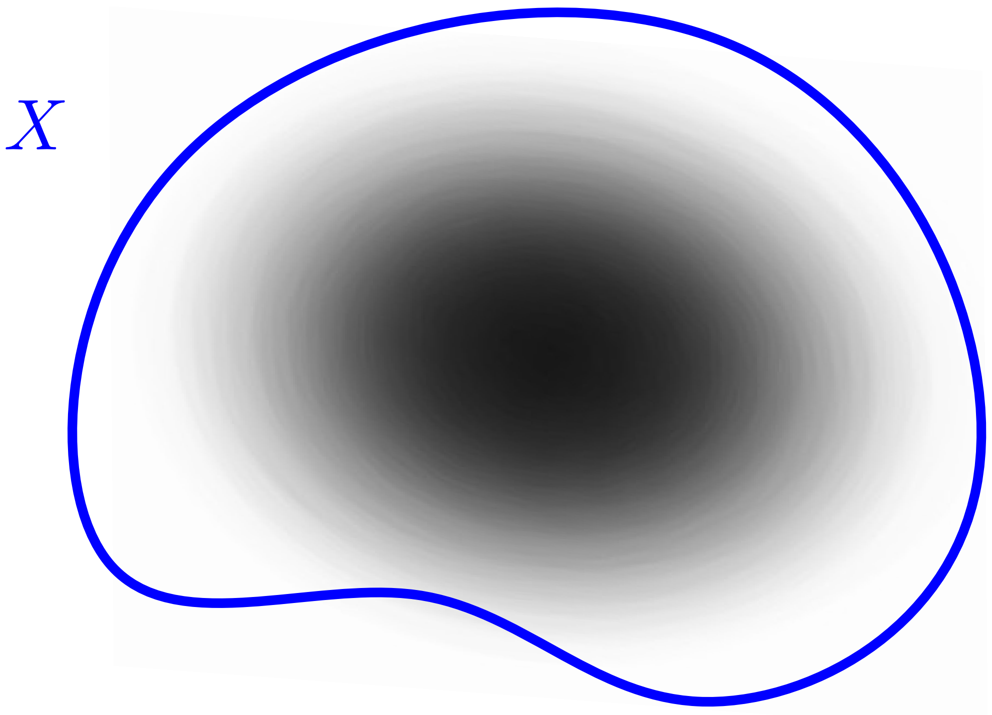

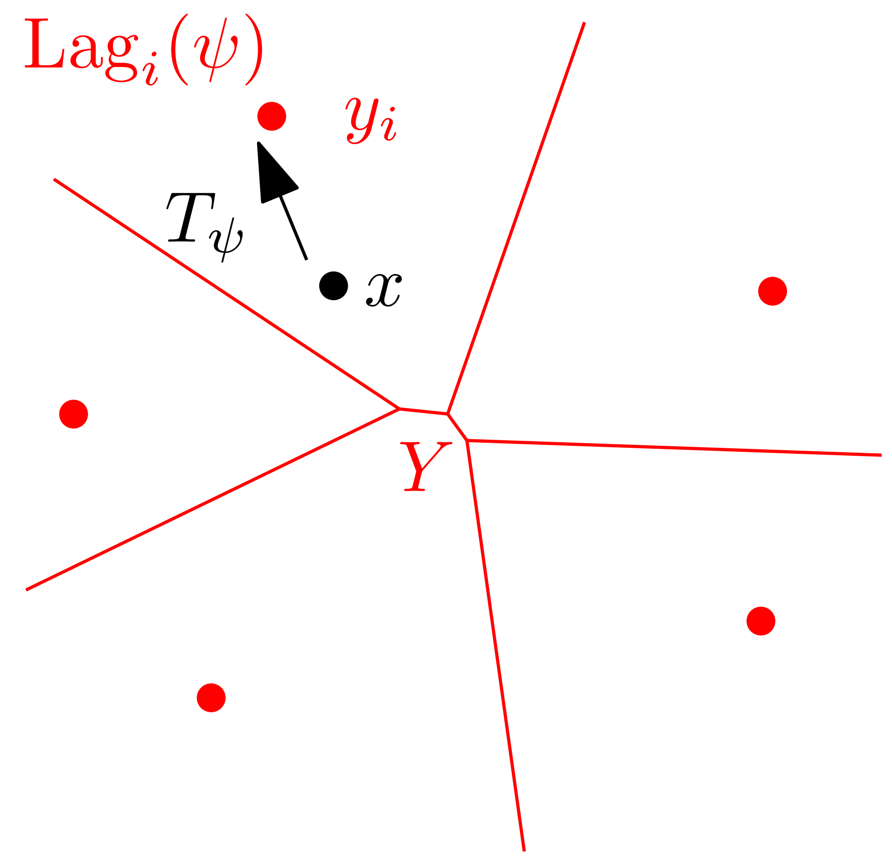

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Definition: For $\psi \in \mathbb{R}^N$, we define $T_\psi : x \in X \mapsto \mathrm{argmin}_{1 \leq i \leq N}(c(x, y_i) + \psi_i) \in Y$

Discrete Monge-Ampère equation

Recall: $\Phi(\psi) = \sum_{i=1}^N \int_{\mathrm{Lag}_i(\psi)} (c(x, y_i) + \psi_i) \,\mathrm{d}\,\mu(x) - \sum_{i=1}^N \nu_i \psi_i$

Theorem: Regularity of Φ

If μ is AC and verifies the (Neg) condition, then Φ is concave and $\mathcal{C}^1$ and

$$\frac{\partial \Phi}{\partial \psi_i}(\psi) = G_i(\psi) - \nu_i \text{ where } G_i(\psi) := \mu(\mathrm{Lag}_i(\psi))$$

Corollary: T_ψ is an optimal transport map between μ and ν

$\iff$ ψ is a maximizer of Φ

$\iff$ $\nabla \Phi(\psi) = 0$

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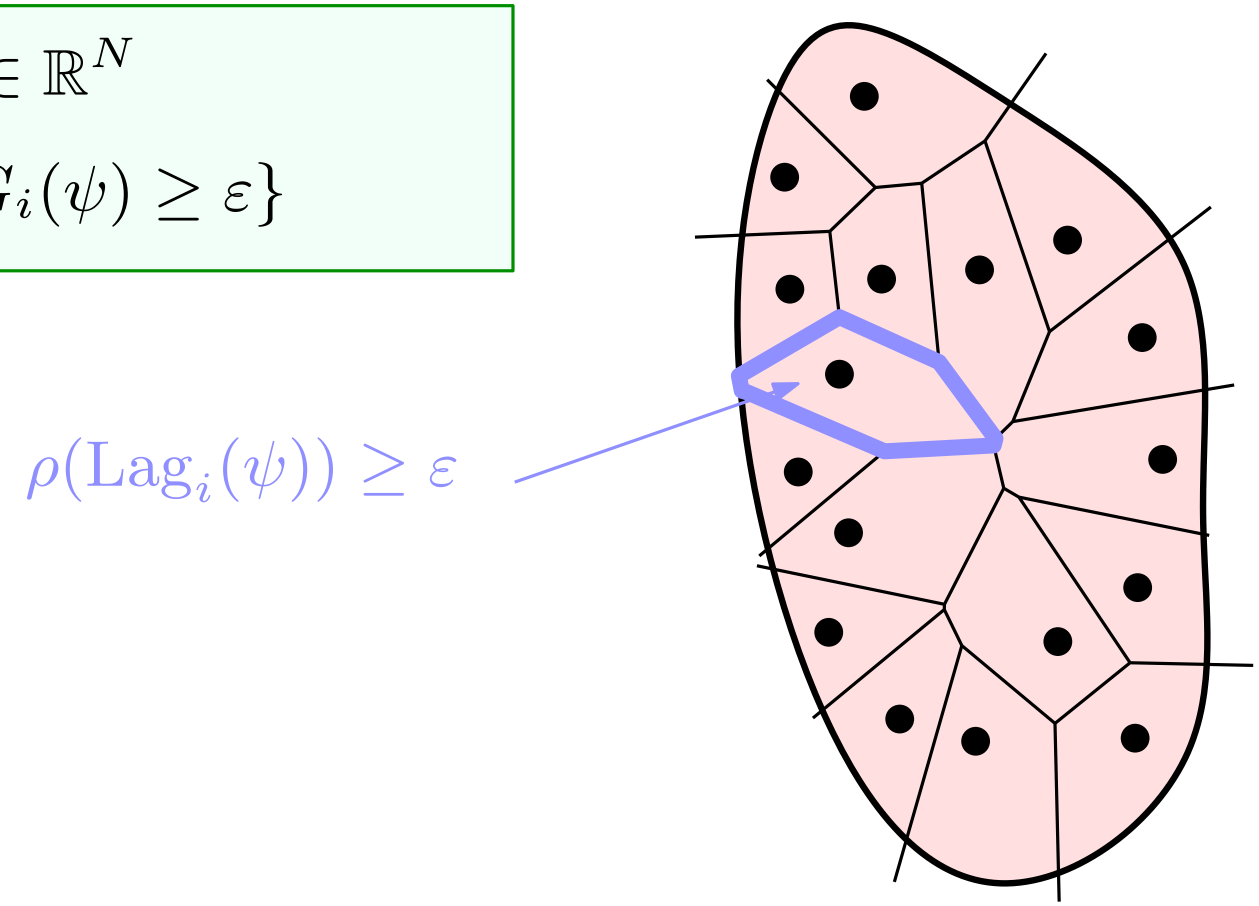
Numerical methods?

Damped Newton Algorithm: description

Recall: $G : \psi \in \mathbb{R}^N \mapsto (\mu(\mathrm{Lag}_i(\psi)))_{1 \leq i \leq N} \in \mathbb{R}^N$

Admissible domain: $E_\varepsilon := \{\psi \in \mathbb{R}^N \mid \forall i,\ G_i(\psi) \geq \varepsilon\}$



[Mirebeau '15]

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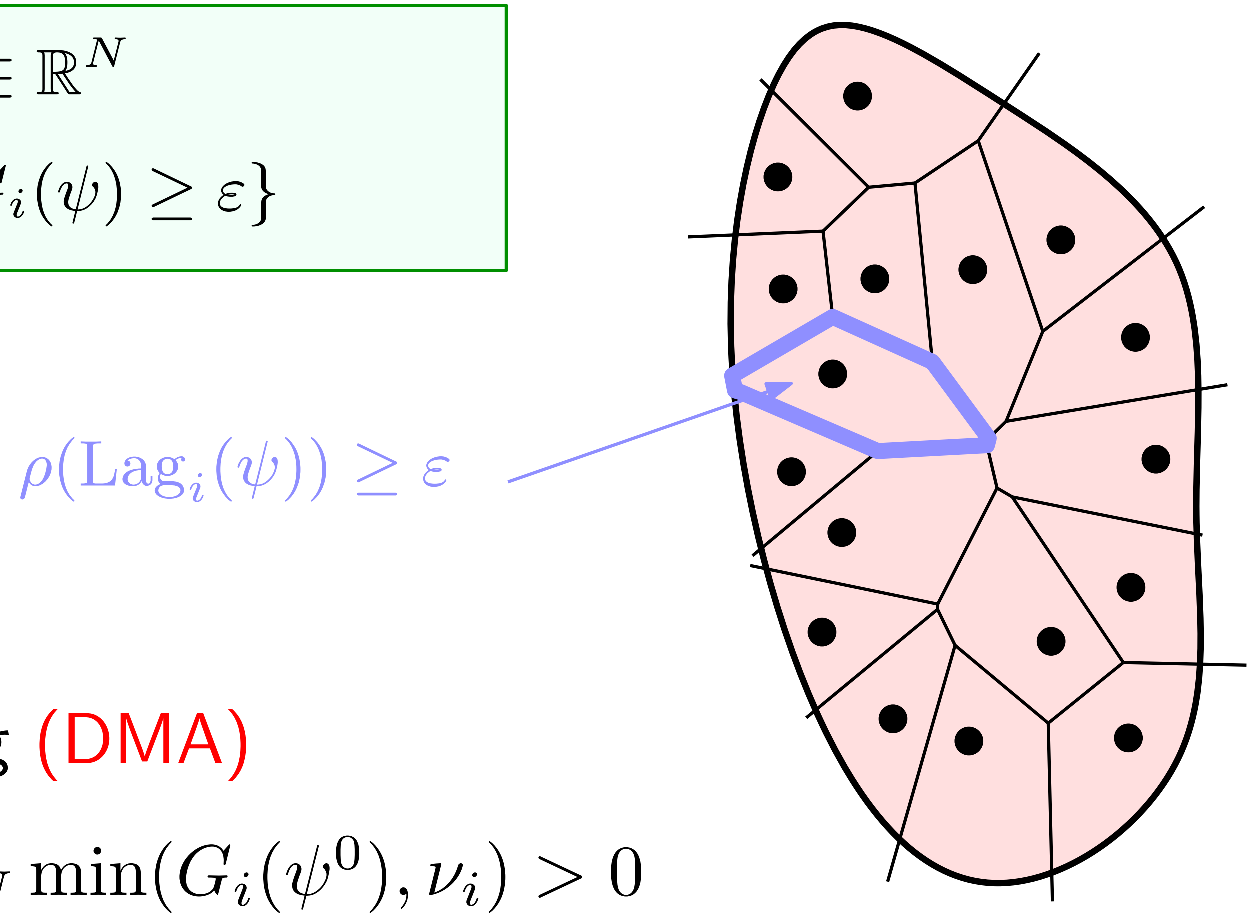

Damped Newton algorithm for solving (DMA)

Input: $\psi^0 \in \mathbb{R}^N$ s.t. $\varepsilon := \frac{1}{2}\min_{1\le i\le N}\min(G_i(\psi^0), \nu_i) > 0$

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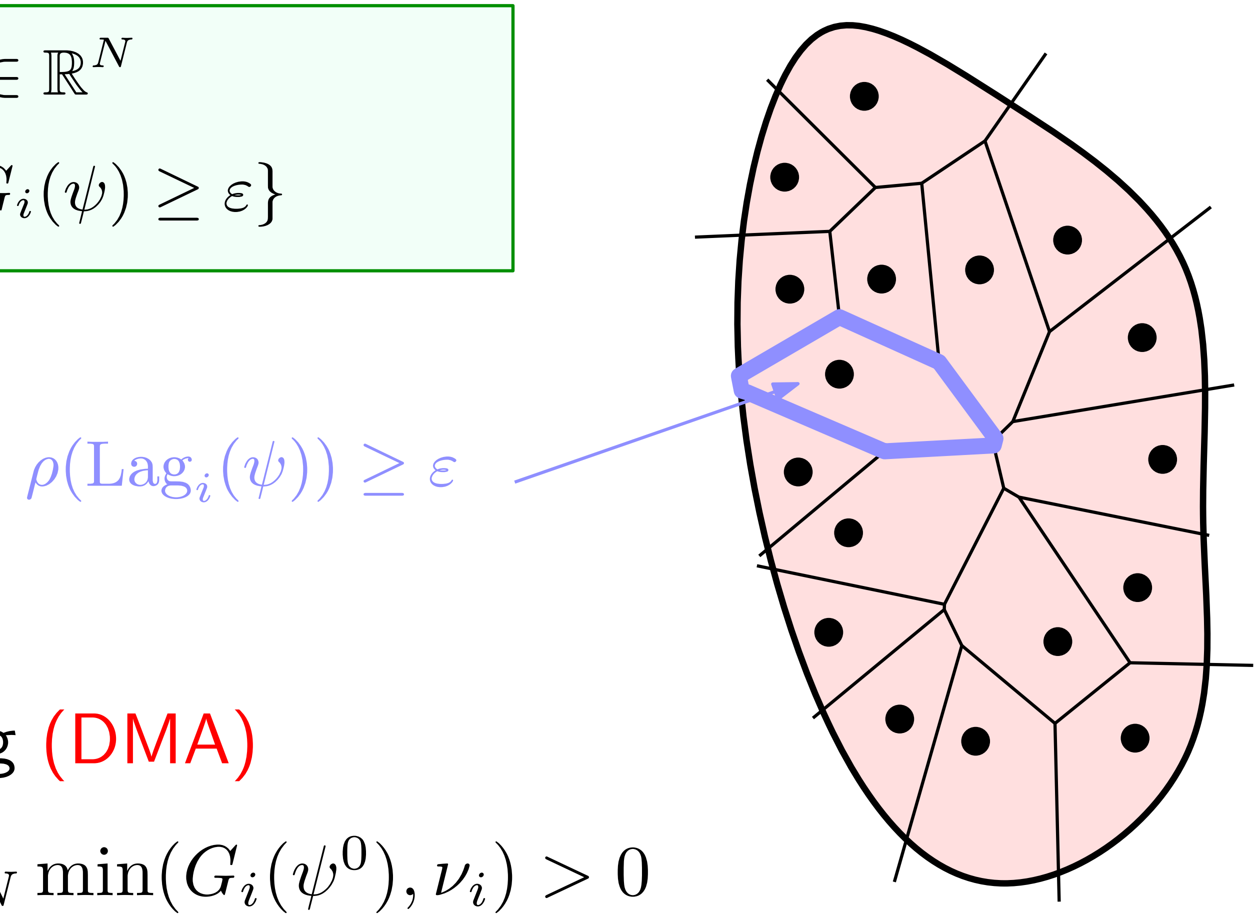

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Loop: $\longrightarrow$ Compute Newton direction: $v^k := -\mathrm{D}G(\psi^k)^+(G(\psi^k) - \nu)$

$\longrightarrow$ Choose ℓ so that $\psi^{k+1} := \psi^k + 2^{-\ell} v_k \in E_\varepsilon$ — Damping

and $\|G(\psi^{k+1}) - \nu\| \leq (1 - 2^{(\ell+1)})\|G(\psi^k) - \nu\|$

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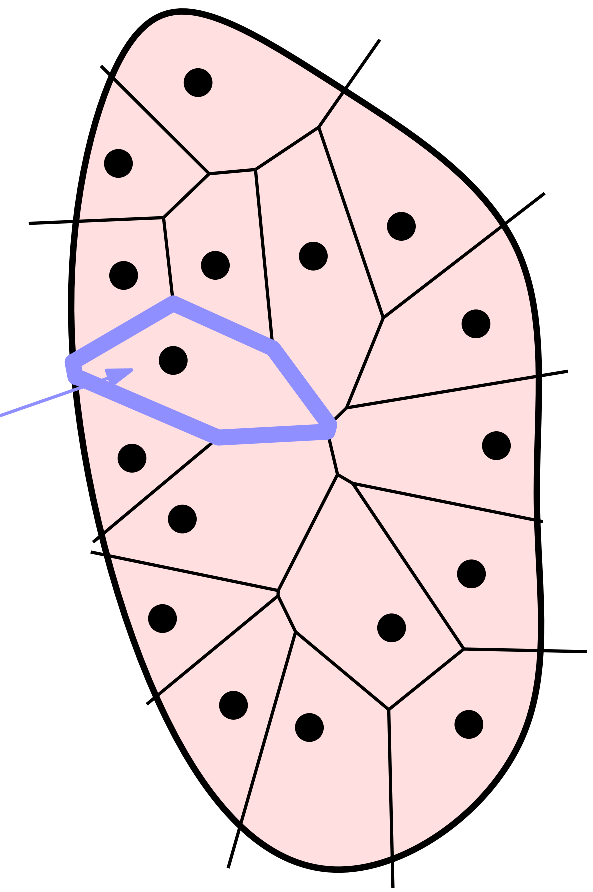
Damped Newton Algorithm: description

Recall: $G : \psi \in \mathbb{R}^N \mapsto (\mu(\mathrm{Lag}_i(\psi)))_{1\leq i\leq N} \in \mathbb{R}^N$

Admissible domain: $E_\varepsilon := \{\psi \in \mathbb{R}^N \mid \forall i,\ G_i(\psi) \geq \varepsilon\}$



Damped Newton algorithm for solving (DMA)

Input: $\psi^0 \in \mathbb{R}^N$ s.t. $\varepsilon := \frac{1}{2}\min_{1\leq i\leq N} \min(G_i(\psi^0), \nu_i) > 0$

Loop: $\longrightarrow$ Compute Newton direction: $v^k := -\mathrm{D}G(\psi^k)^+(G(\psi^k) - \nu)$

$\longrightarrow$ Choose ℓ so that $\psi^{k+1} := \psi^k + 2^{-\ell} v_k \in E_\varepsilon$ — Damping

and $\|G(\psi^{k+1}) - \nu\| \leq (1 - 2^{(\ell+1)})\|G(\psi^k) - \nu\|$

[Mirebeau '15]

$\Longrightarrow$ Convergence when X is a triangulated surface?

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OT between a triangulation and a point cloud

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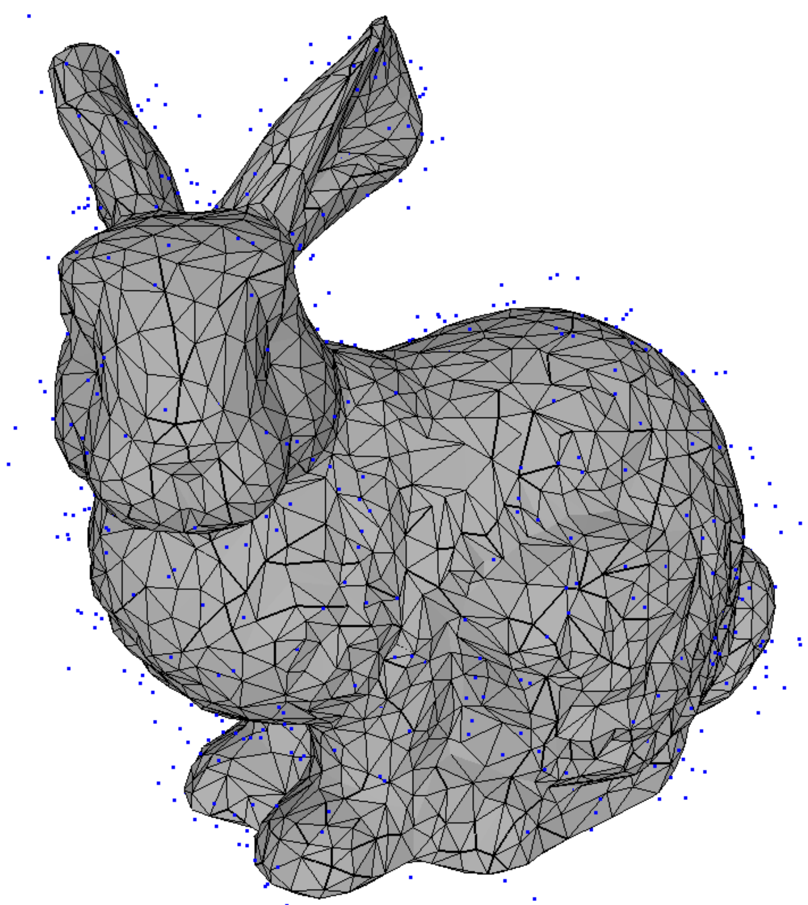
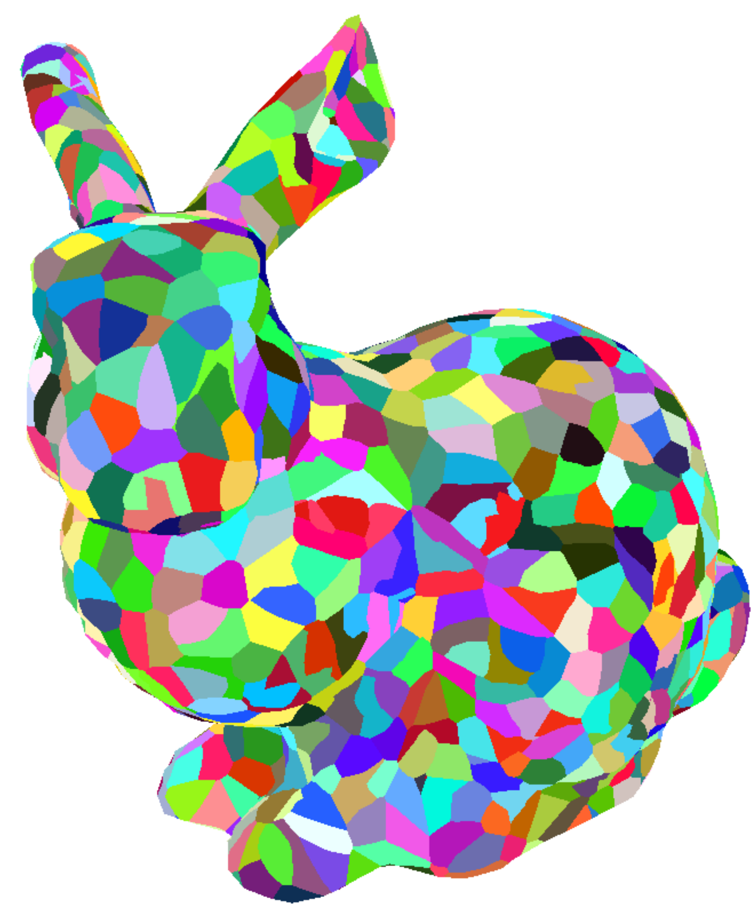
Output:

- Transport plan between μ and ν for *quadratic* cost $\rightsquigarrow$ Laguerre cells $(\mathrm{Lag}_i(\psi))_{1 \leq i \leq N}$





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Solution: use a *genericity* assumption on the point cloud Y and *regularity* on μ

Main theorem

[Mérigot, M., Thibert, SIIMS '18]

Theorem:

Assume μ is a regular simplicial measure

$y_1, \cdots, y_N$ are in generic position

Then the damped Newton method converges with linear rate globally i.e.

$$\|G(\psi^k) - \nu\| \leq (1 - \frac{\tau^*}{2})^k \|G(\psi^0) - \nu\| \text{ where } \tau^* \in]0,1]$$

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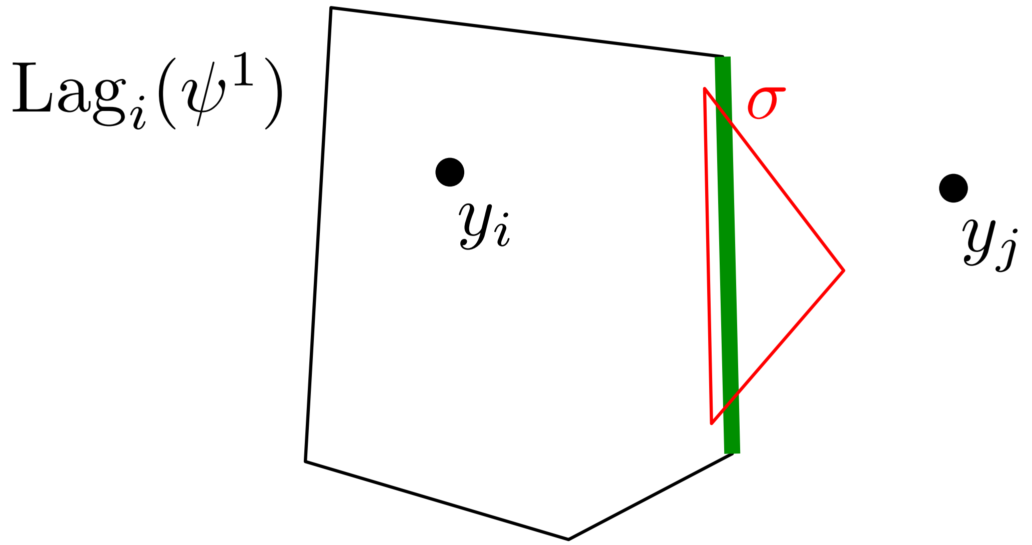
Genericity position: example of non-generic case, edge of $\sigma \perp (y_i y_j)$



$$\frac{\partial G_i}{\partial \psi_j}(\psi^1) \propto \mu(\partial \, \mathrm{Lag}_i(\psi^1) \cap \sigma) > 0$$

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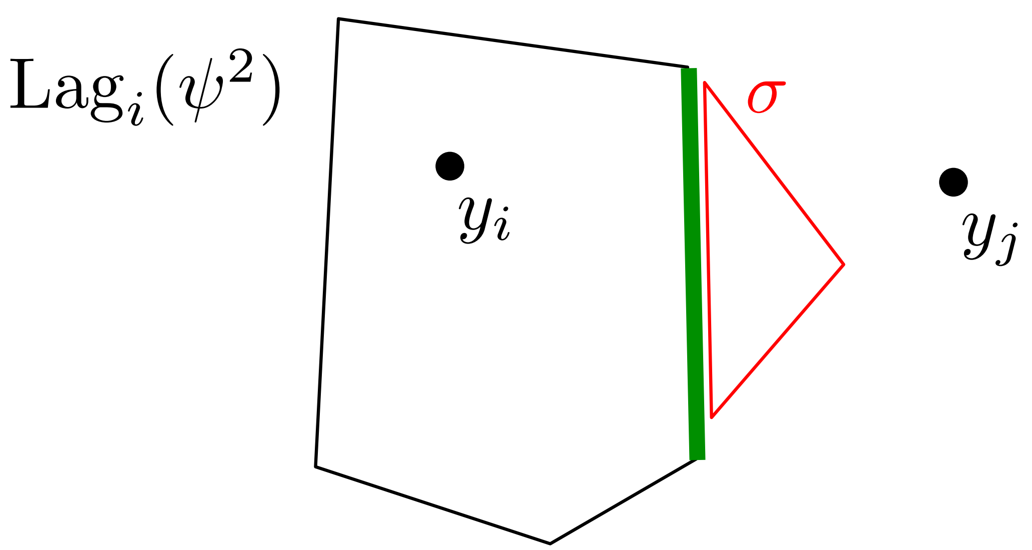
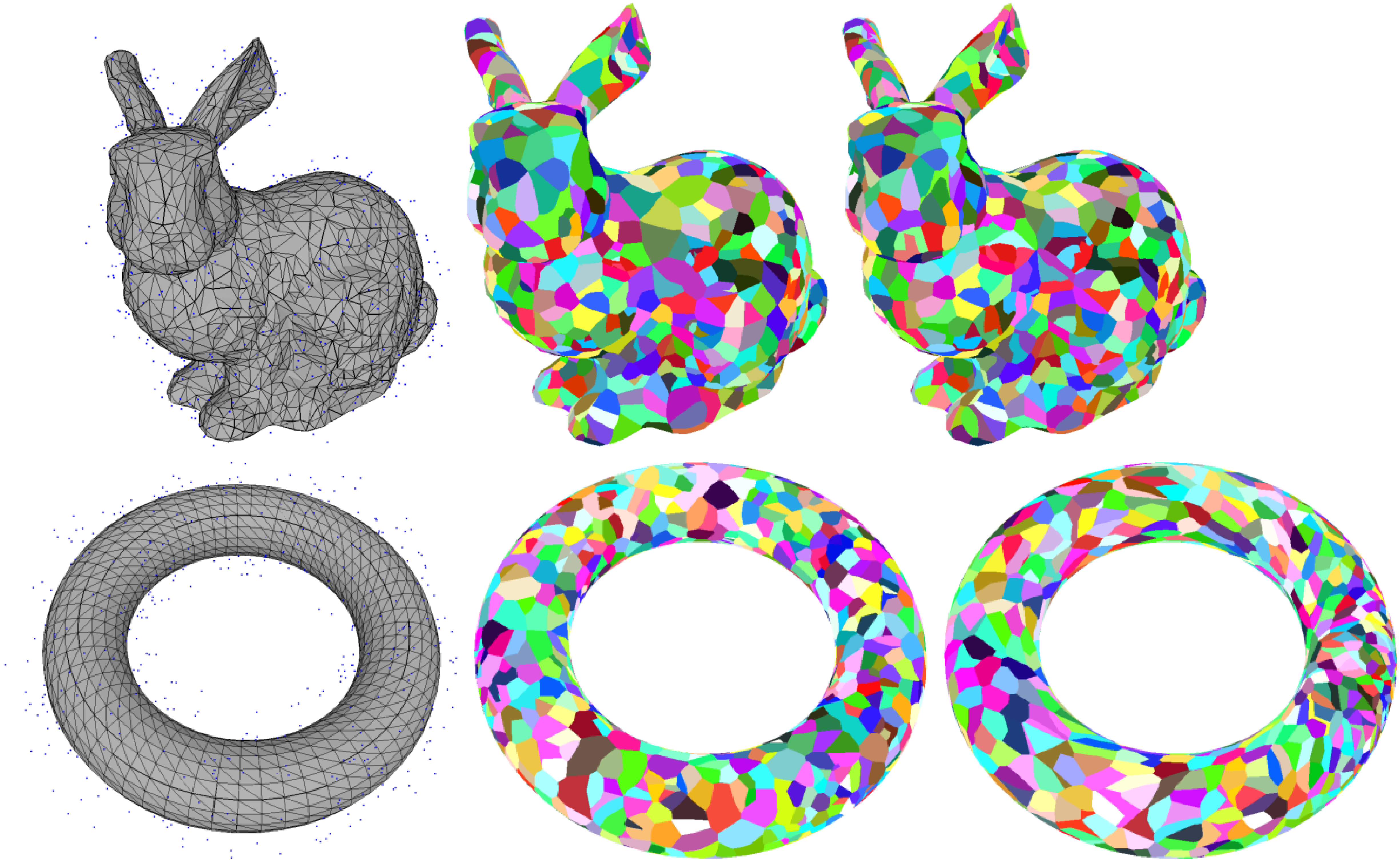


$$\frac{\partial G_i}{\partial \psi_j}(\psi^2) \propto \mu(\partial \,\mathrm{Lag}_i(\psi^2) \cap \sigma) = 0$$

$\implies G$ not $\mathcal{C}^1$

Numerical results

Optimal transport for a **uniform** source



Initial: ψ^0 **Final**

Numerical results

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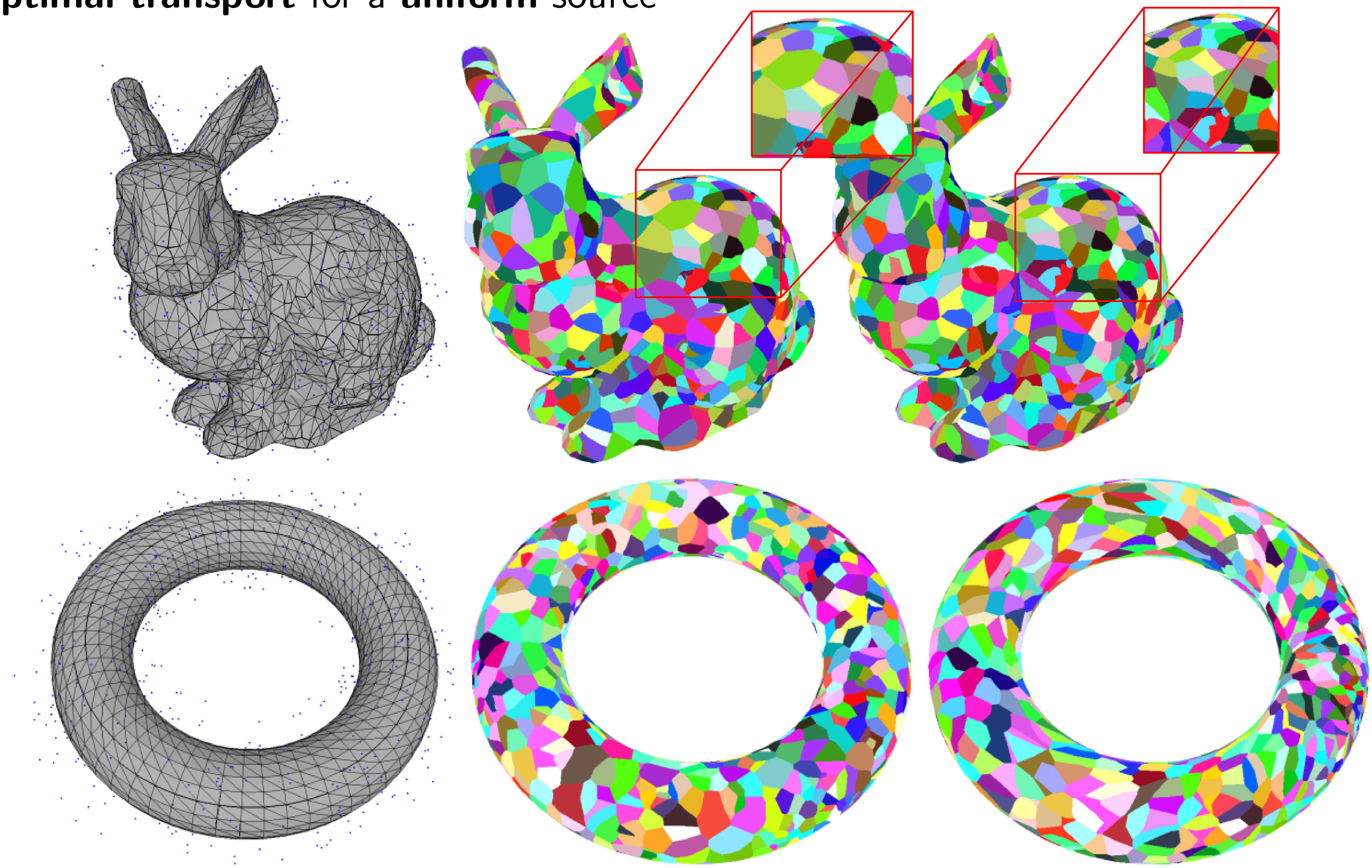
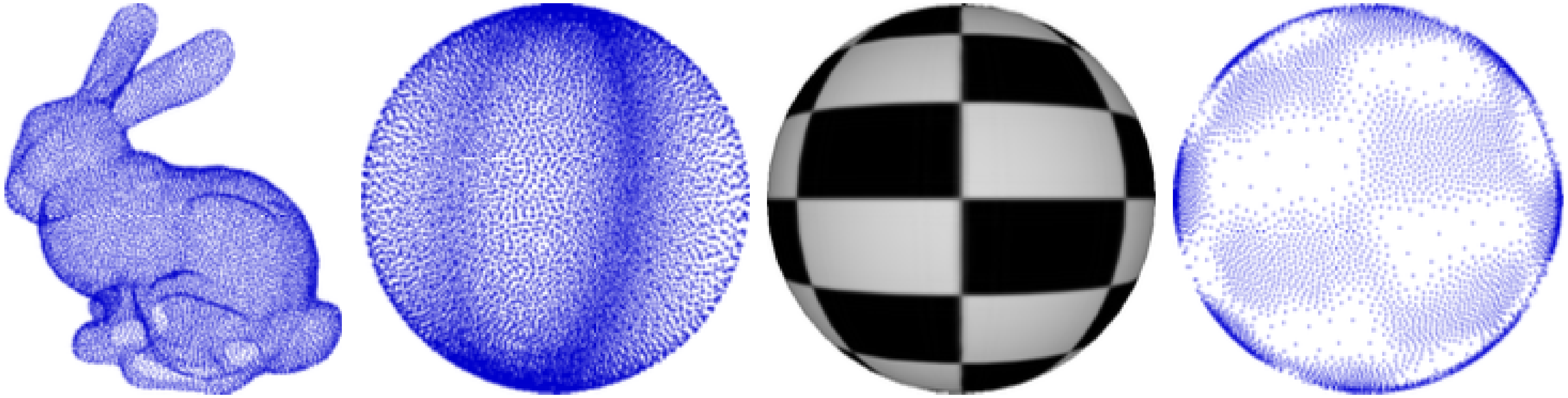


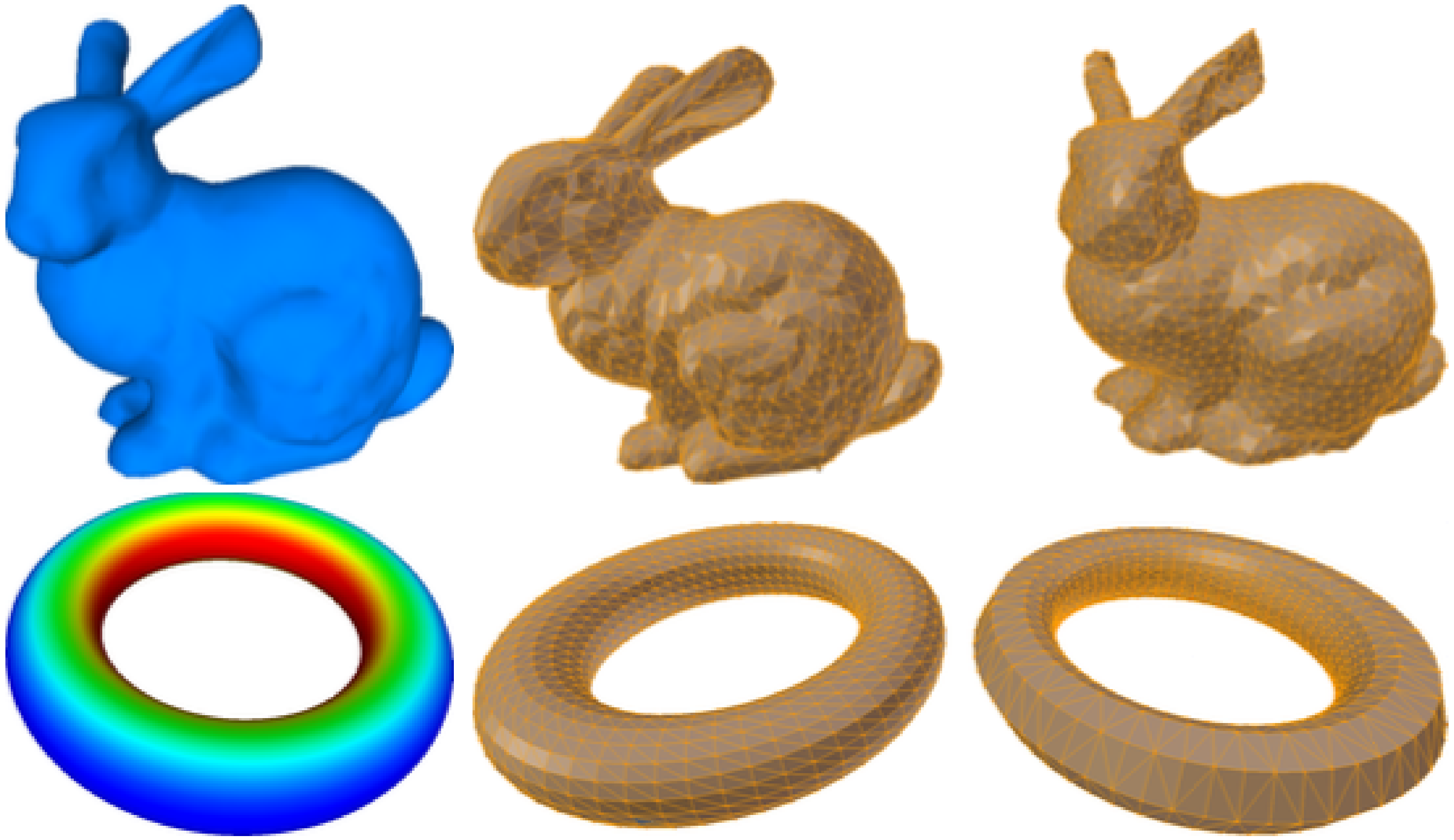
Initial: ψ^0 **Final**

Numerical results

- Optimal quantization of a probability measure on a triangulated surface



- Remeshing with respect to a density μ (uniform, mean curvature)



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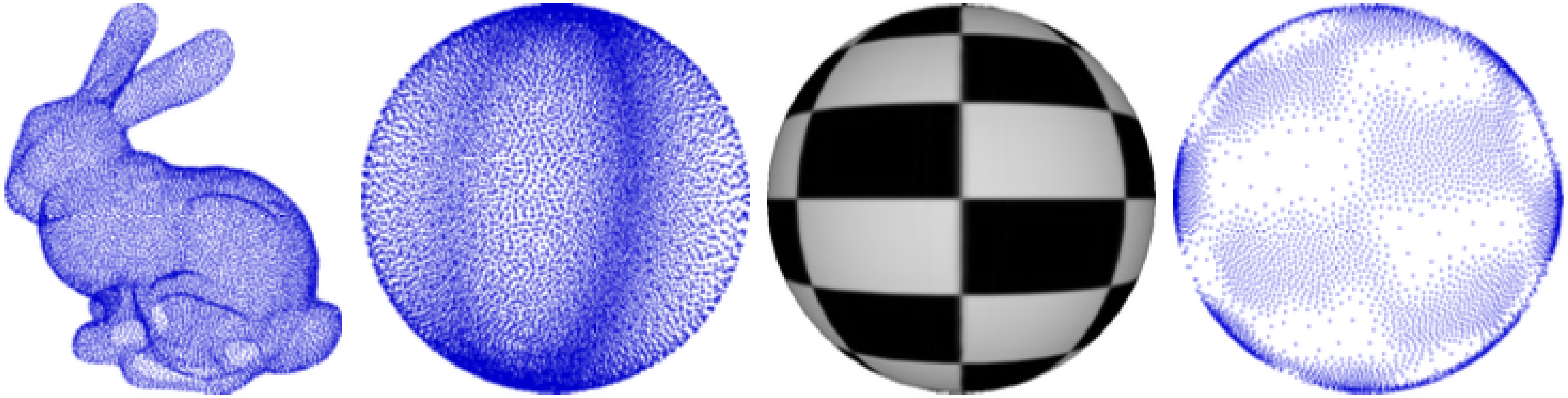

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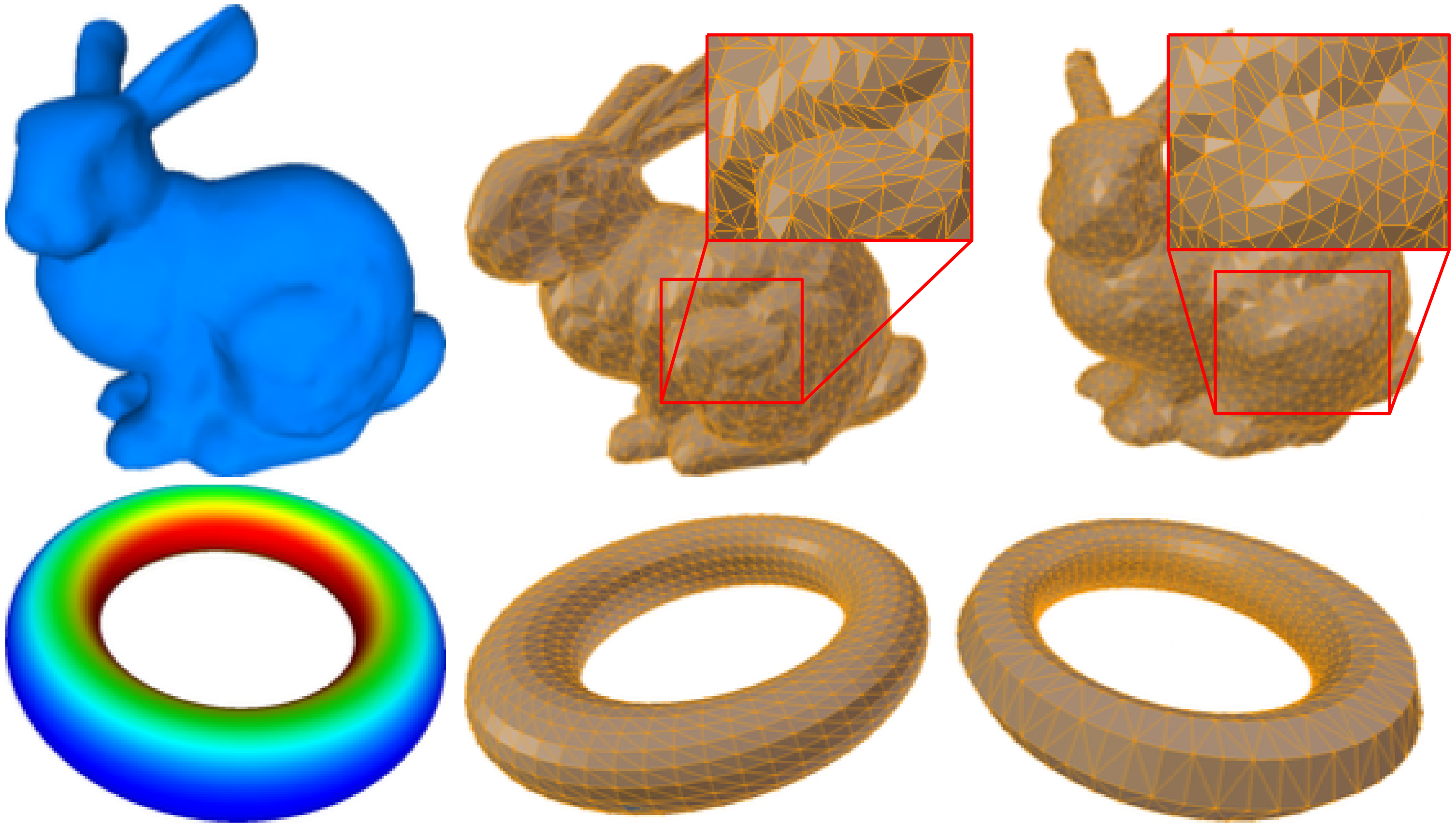

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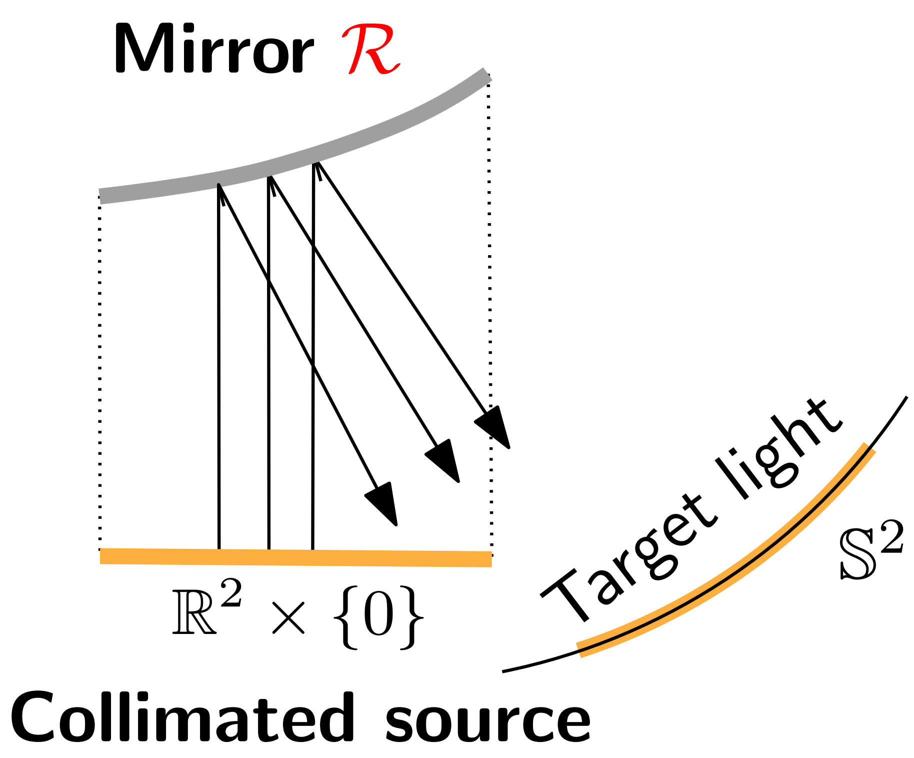
Four non-imaging optics problems



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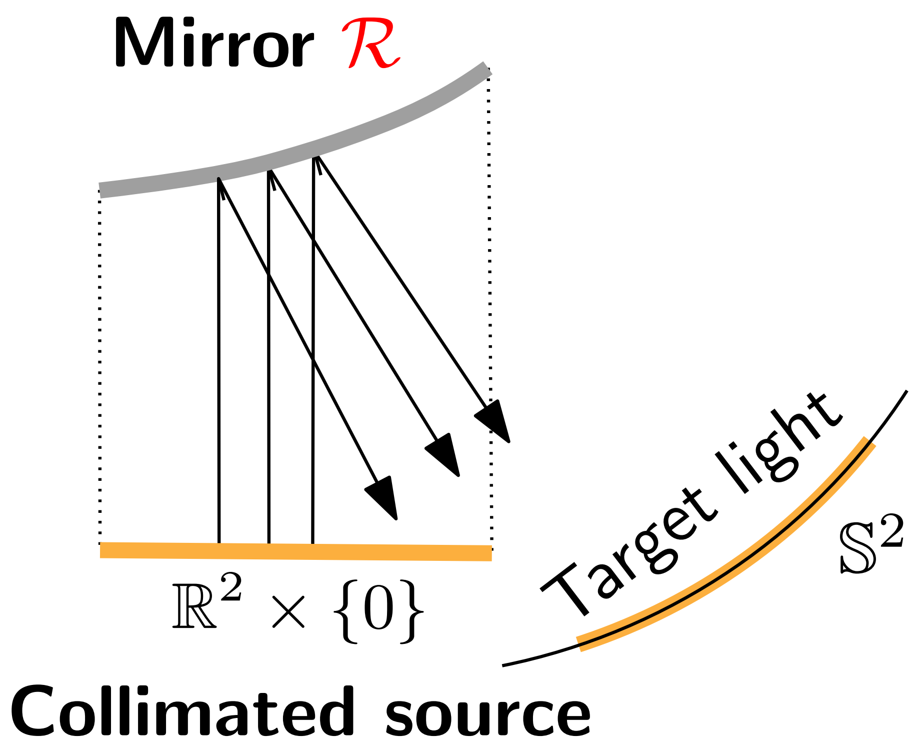

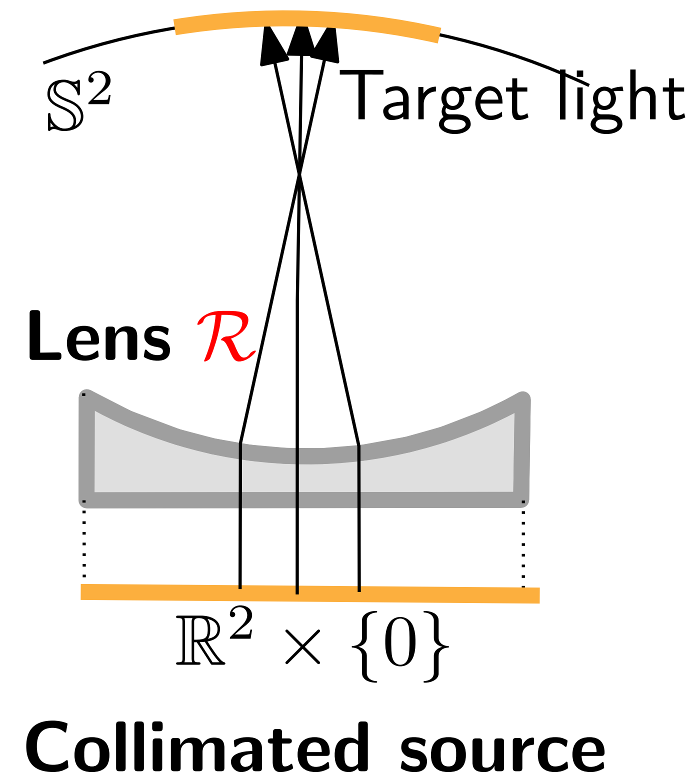

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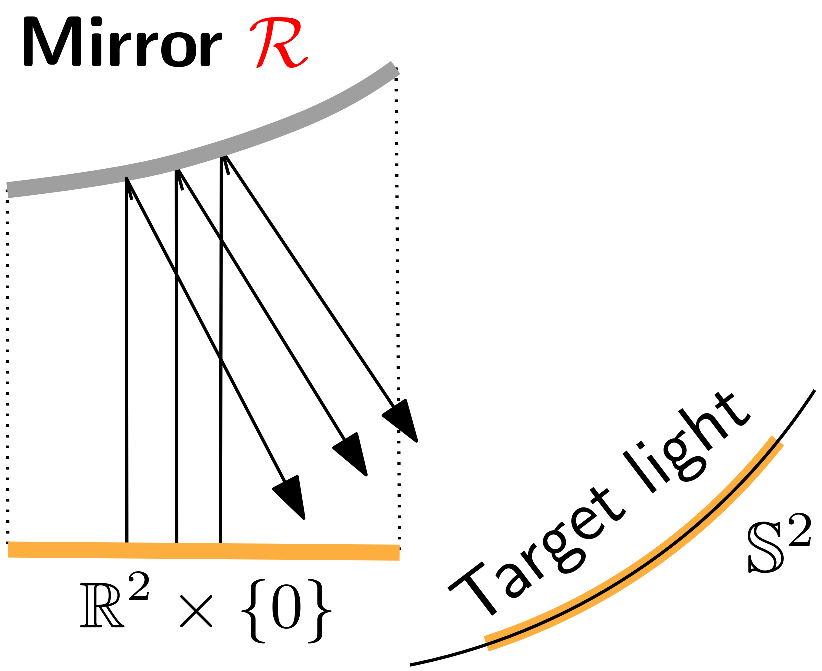
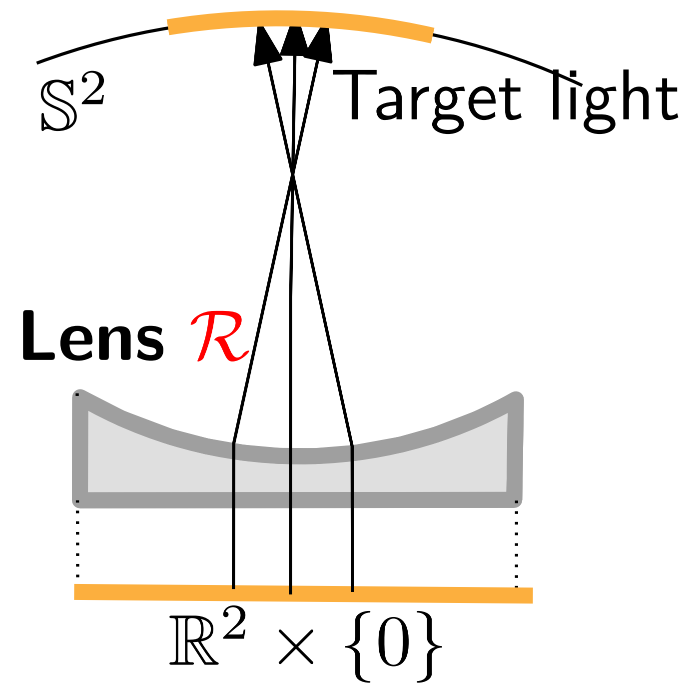


Collimated source



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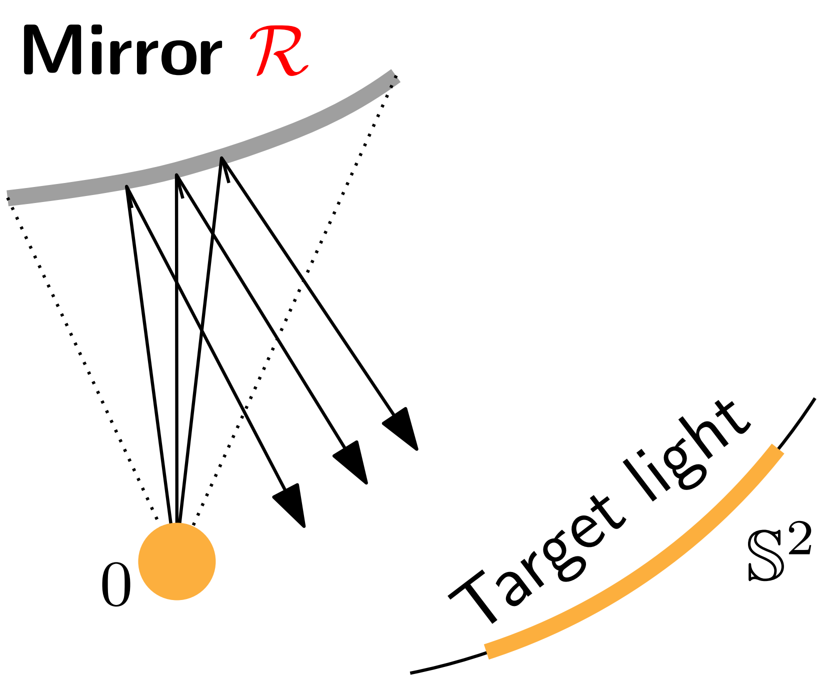
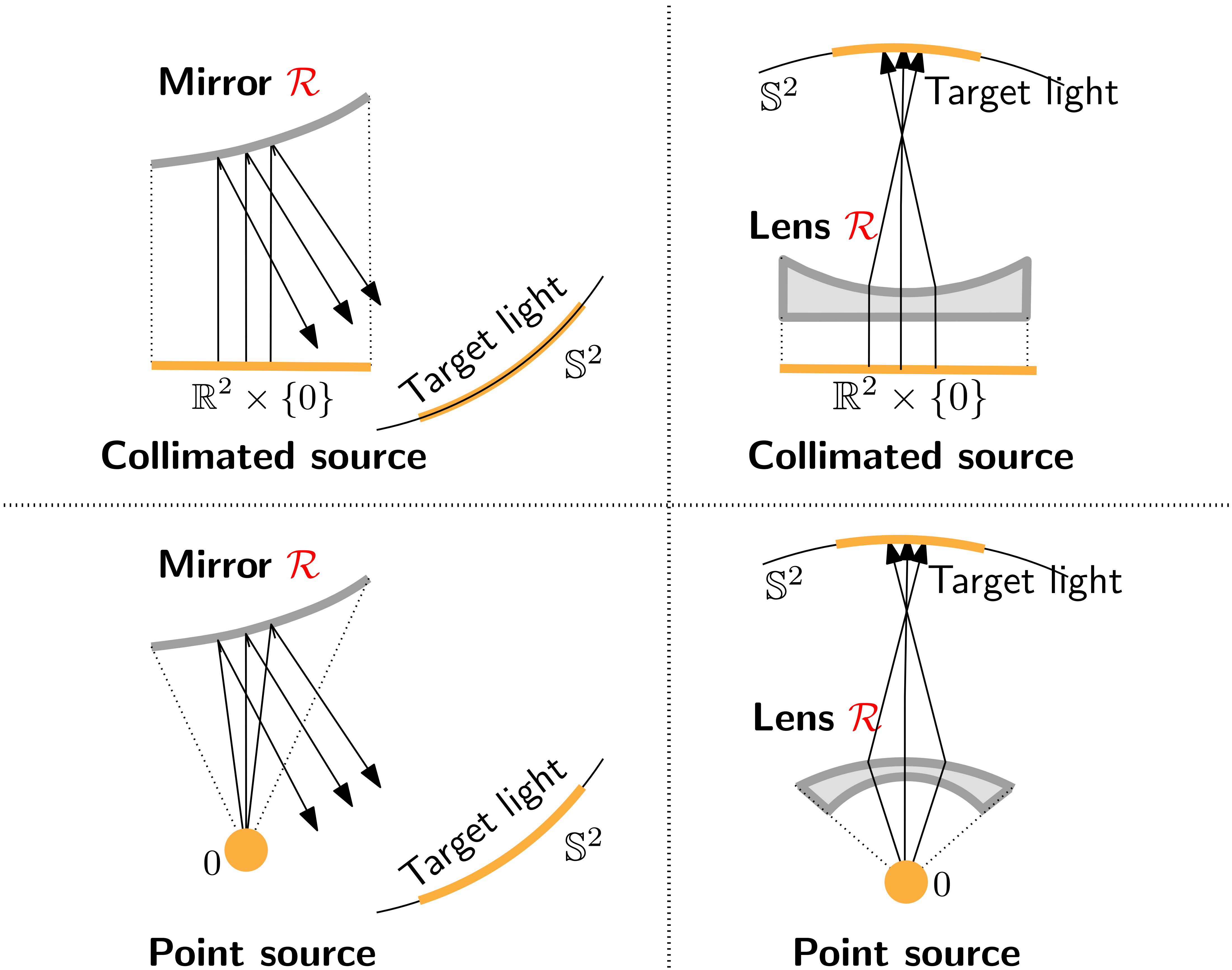


Point source

Four non-imaging optics problems



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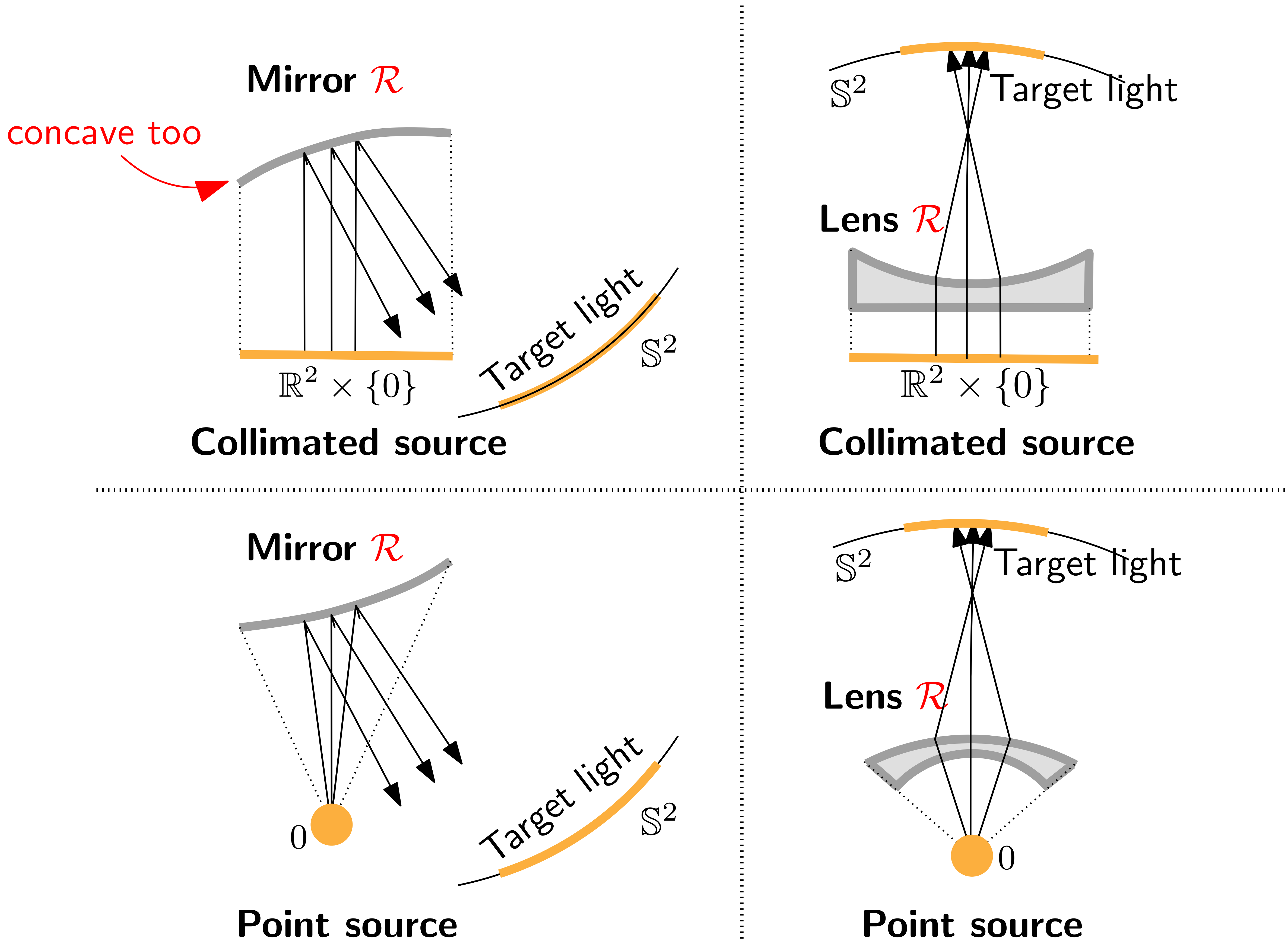

Mirror design for a collimated source

Input: collimated light source μ and target light **at infinity** $\nu = \sum_i \nu_i \delta_{y_i}$

Goal: design a **convex** mirror $\mathcal{R}$ that sends μ to ν

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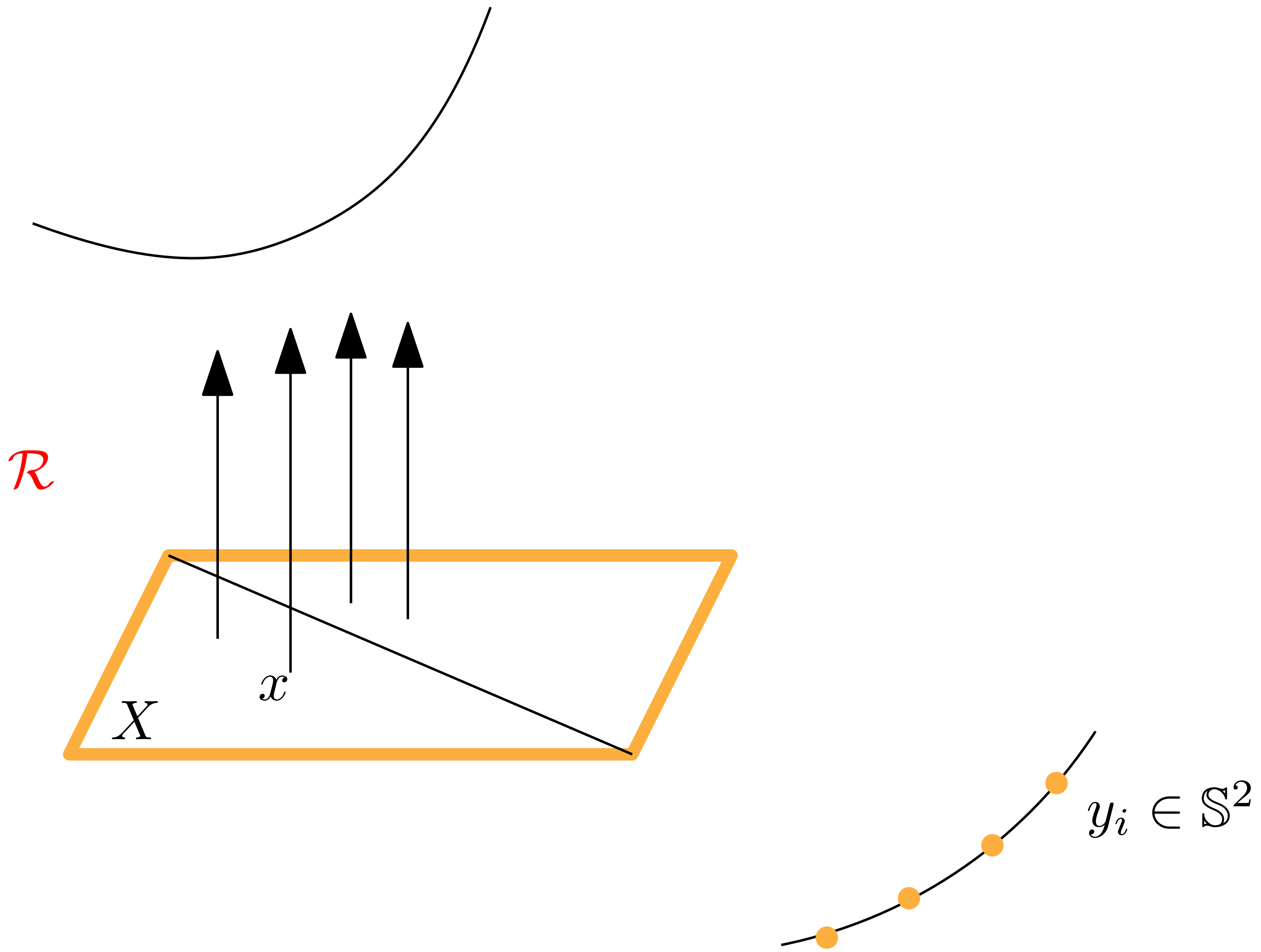
1. **Discretization:** μ supported on a triangulation X and ν on a point cloud Y



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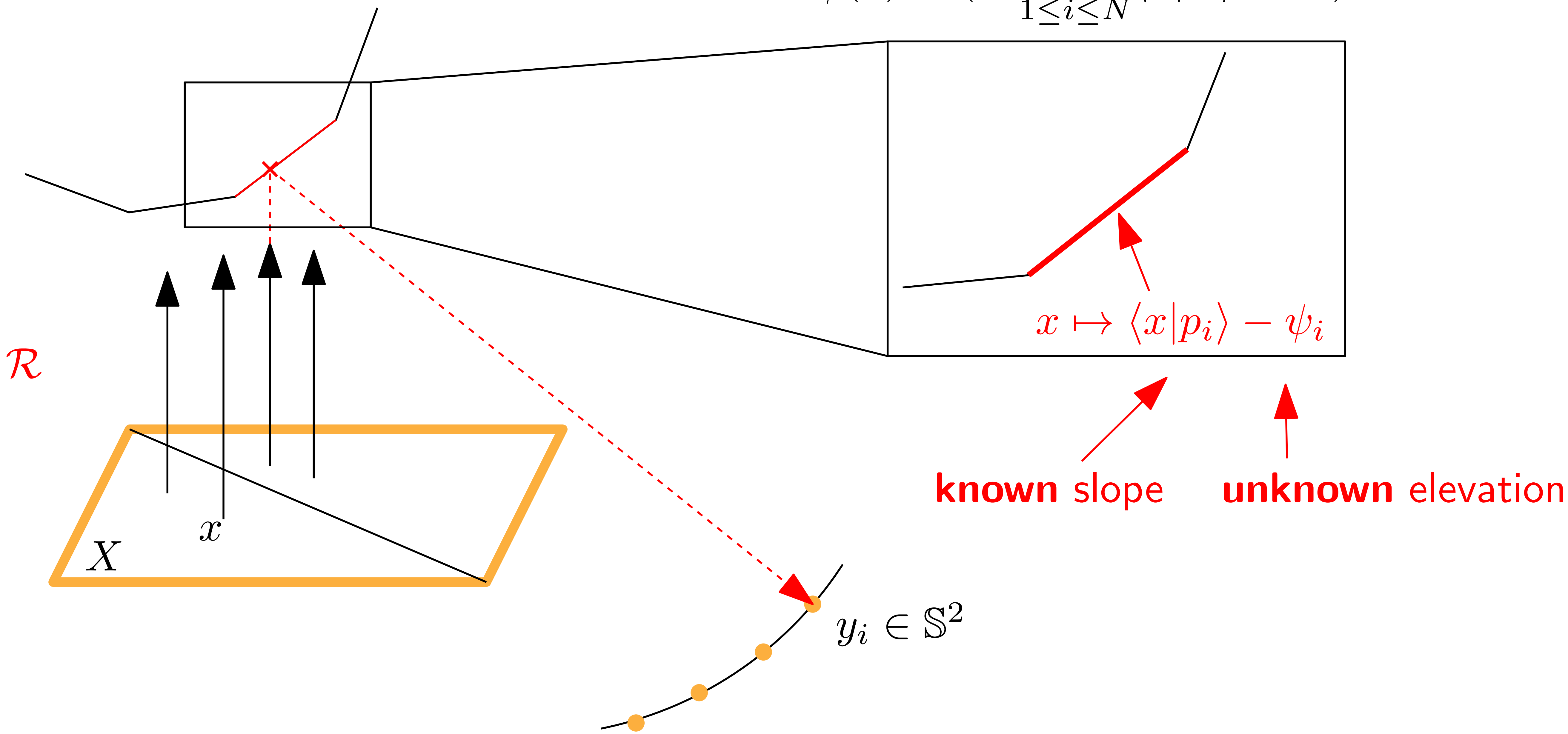
2. $\mathcal{R}$ is convex and can be parametrized by $\mathcal{R}_\psi(x) = (x, \max_{1 \leq i \leq N} \langle x | p_i \rangle - \psi_i)$



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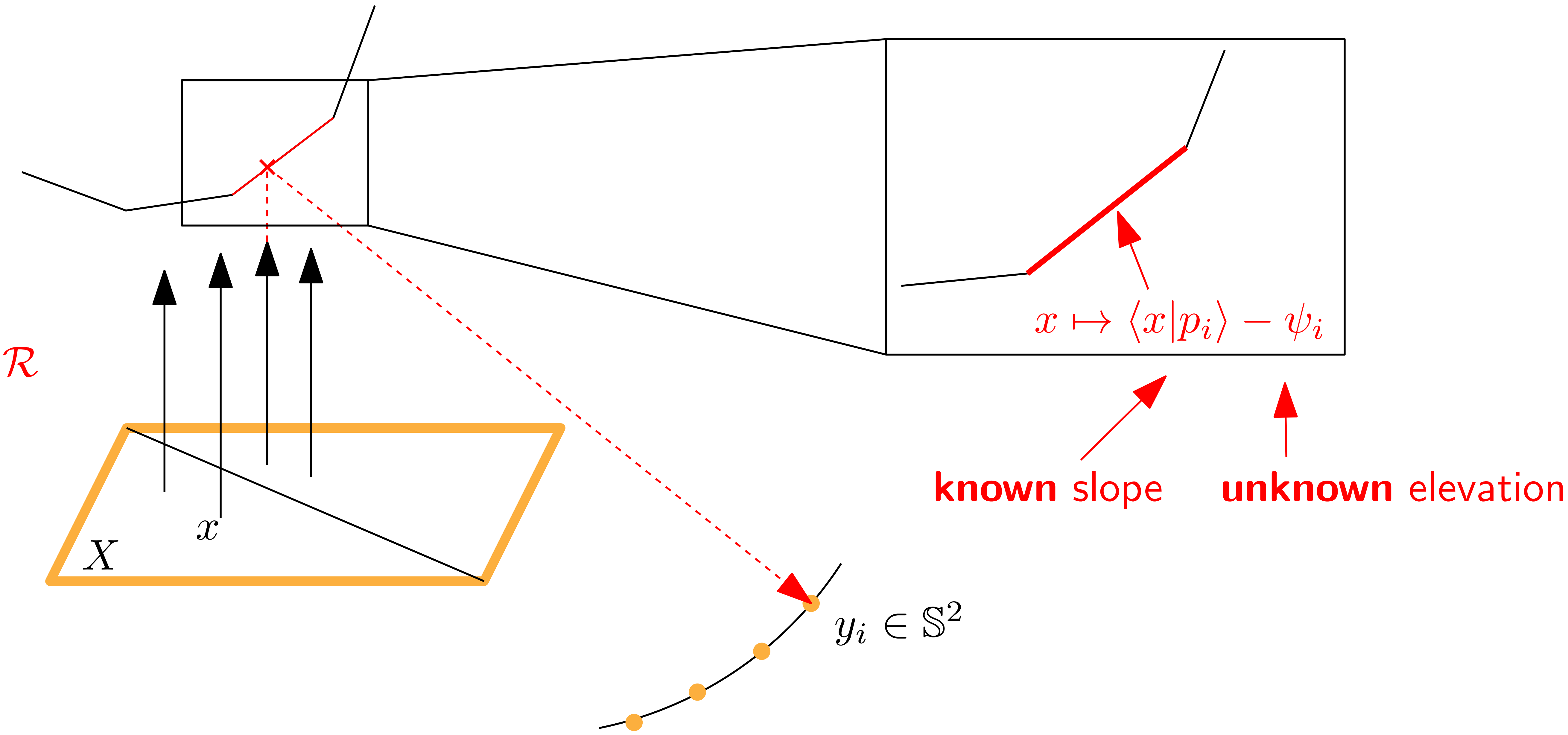
3. $\mathrm{V}_i(\psi) = \{x \in X \mid x$ reflected towards $y_i\}$ and $G_i(\psi) = \mu(\mathrm{V}_i(\psi))$



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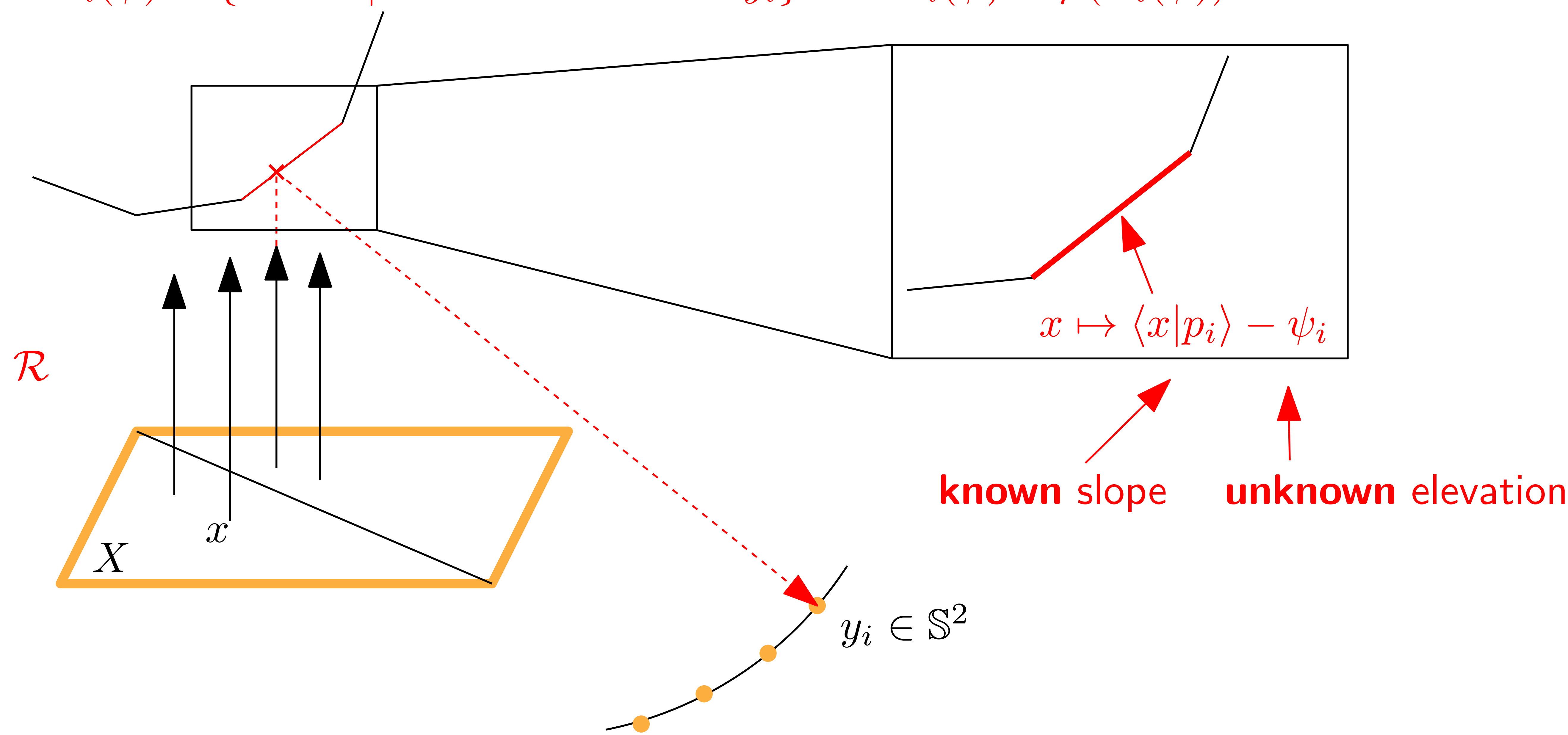

Find $\psi \in \mathbb{R}^N$ such that $\forall i \in \{1, \ldots, N\},\ G_i(\psi) = \nu_i$ (LEC)

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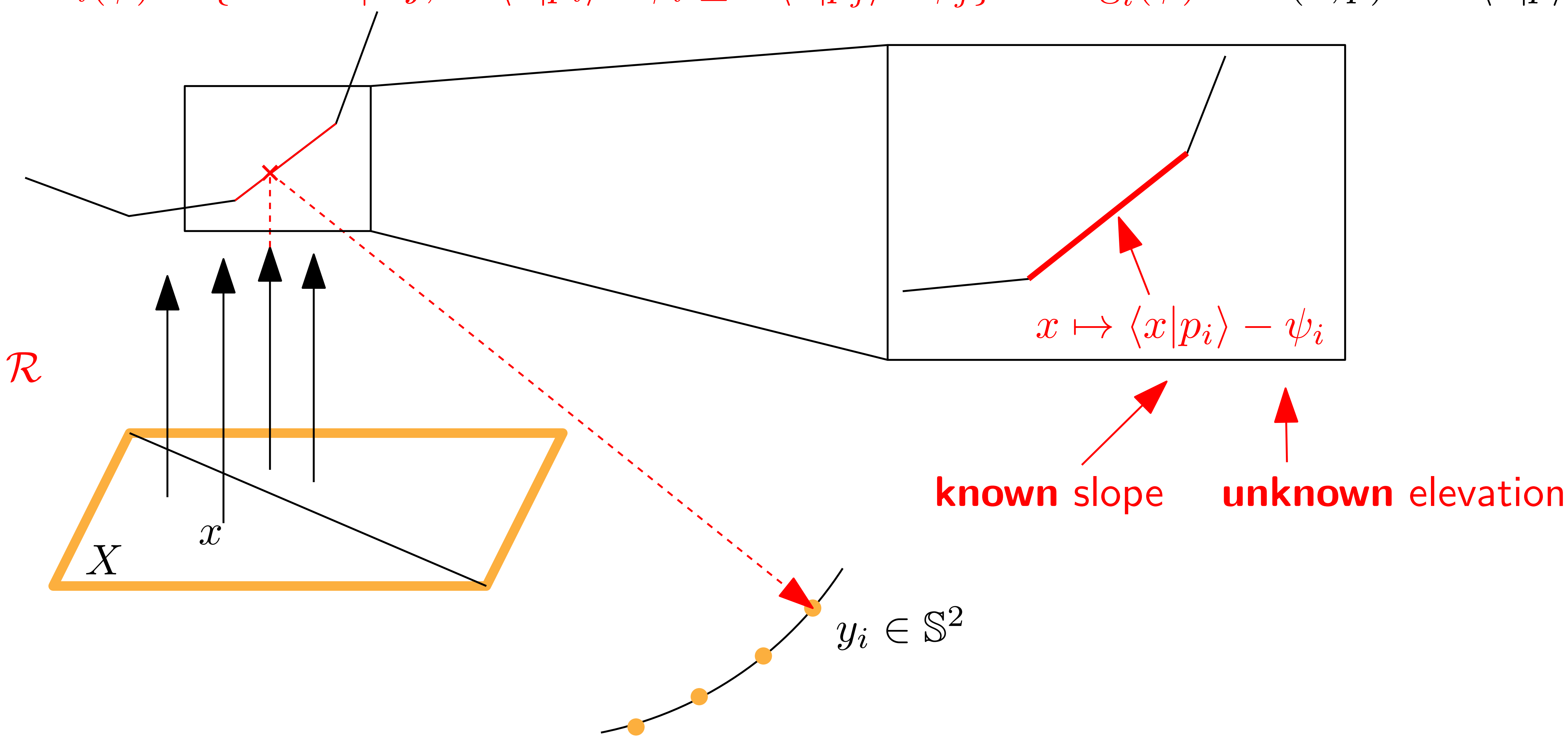
4. $\mathrm{V}_i(\psi) = \{x \in X \mid \forall j,\ -\langle x|p_i\rangle + \psi_i \leq -\langle x|p_j\rangle + \psi_j\} = \mathrm{Lag}_i(\psi)$ for $c(x,p) = -\langle x|p\rangle$



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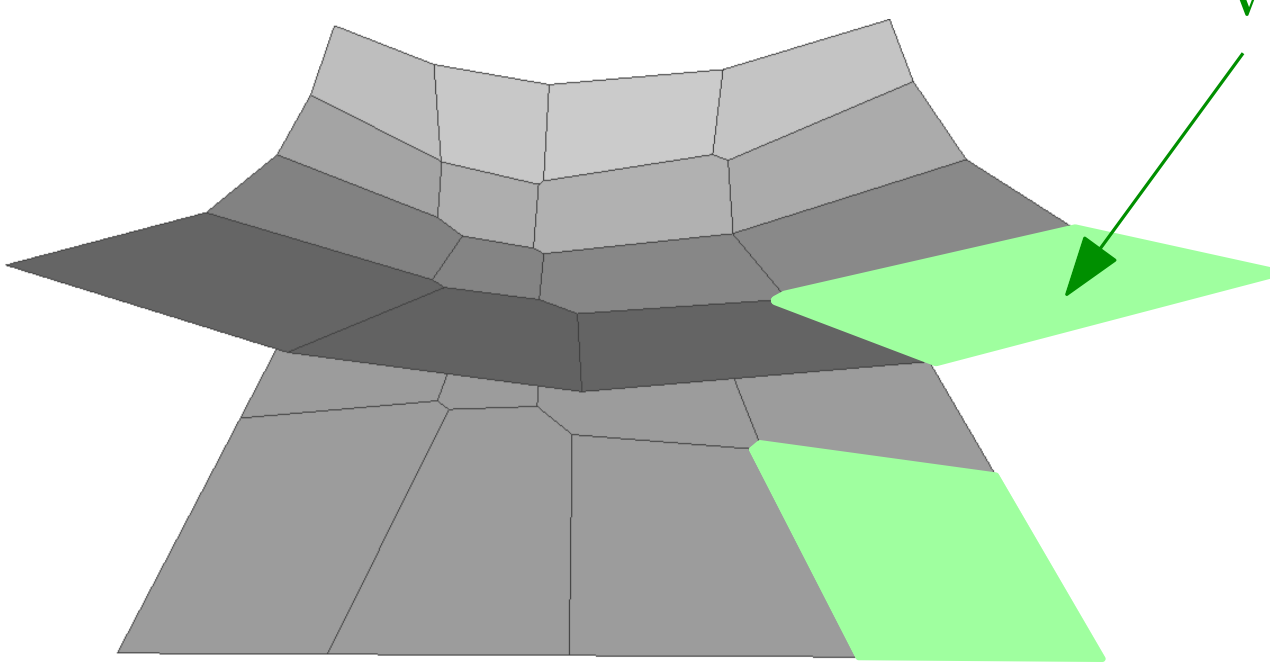
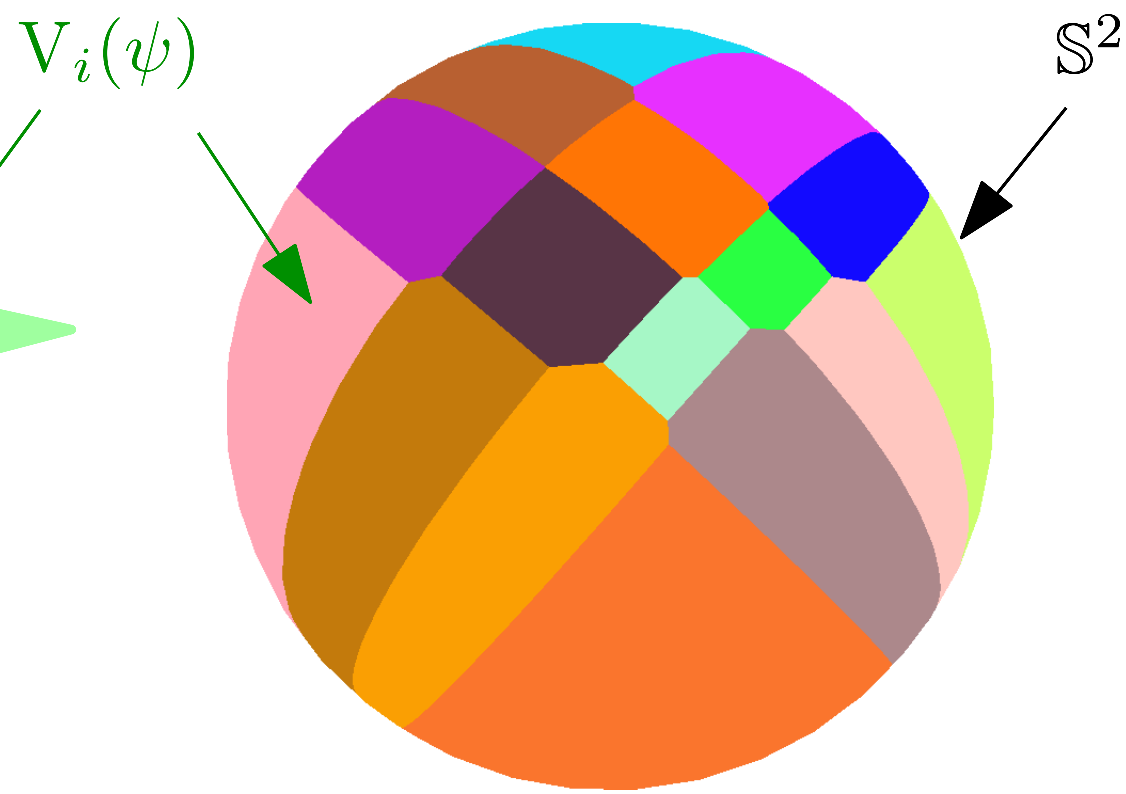
Light Energy Conservation equation

Goal: solve (LEC) where

- $\mathrm{V}_i(\psi)$ is a *Laguerre* cell $\implies$ optimal transport problem for some cost c
- $X \subset \mathbb{R}^2 \times \{0\}$ for collimated lights and $X \subset \mathbb{S}^2$ for point lights
- $\mathcal{R}_\psi$ is a parametrization of the component
  - **piecewise affine** function for mirror & lens / collimated light
  - **pieces of paraboloids** for mirror / point light
  - **pieces of ellipsoids** for lens / point light



Mirror / collimated light



Mirror / point light

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Generic algorithm

Algorithm: Mirror / lens construction

Input A light source X, μ

A target light Y, ν

A tolerance $\eta > 0$

A parametrization function $\psi \mapsto \mathcal{R}_\psi$

A transformation function $\tau : \mathrm{Lag} \mapsto X \cap \mathrm{Pow}$

depends on the optical design problem

Output A triangulation $\mathcal{R}_T$ of a mirror or lens $\mathcal{R}$

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$\mathcal{R}_T \leftarrow \texttt{SURFACE\_CONSTRUCTION}(\psi, \mathcal{R}_\psi)$

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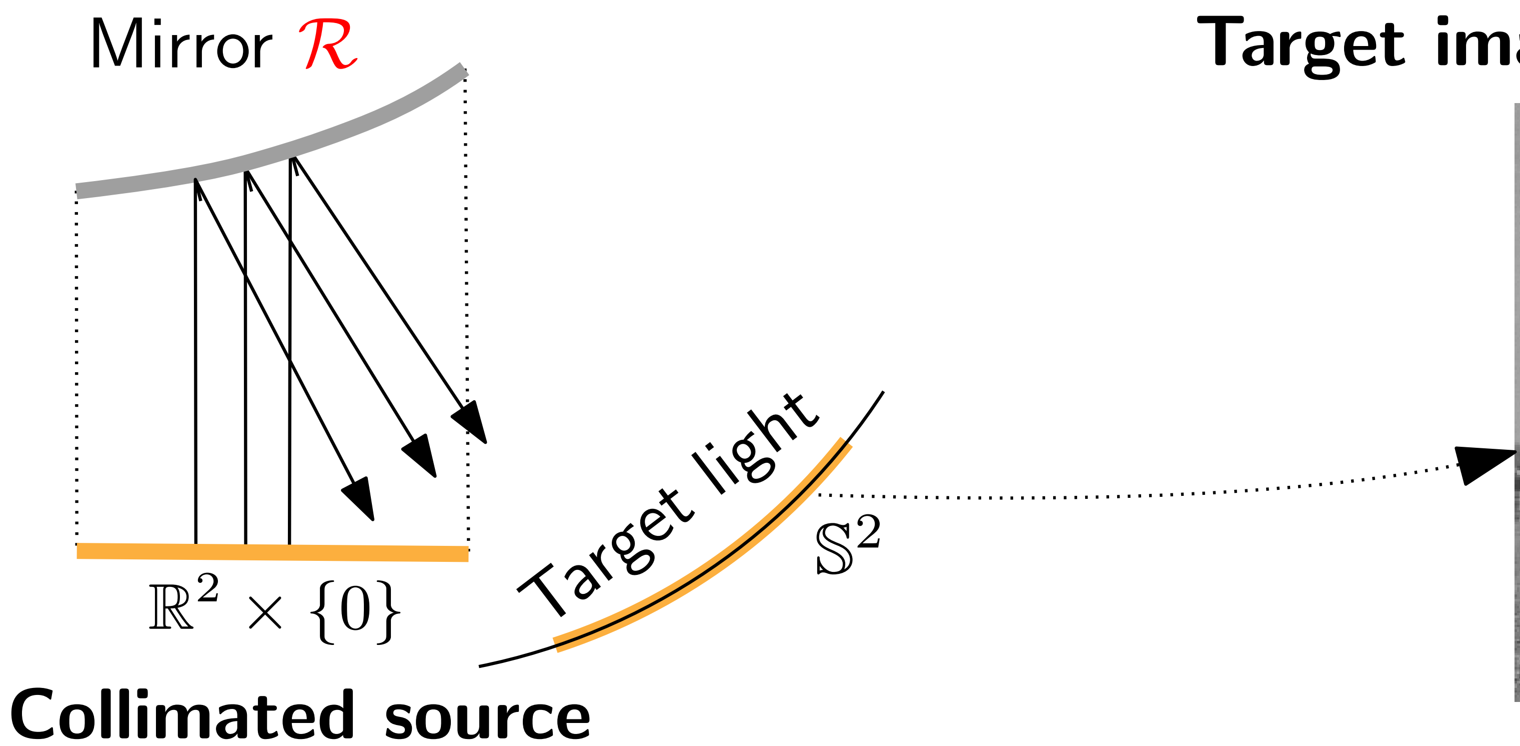
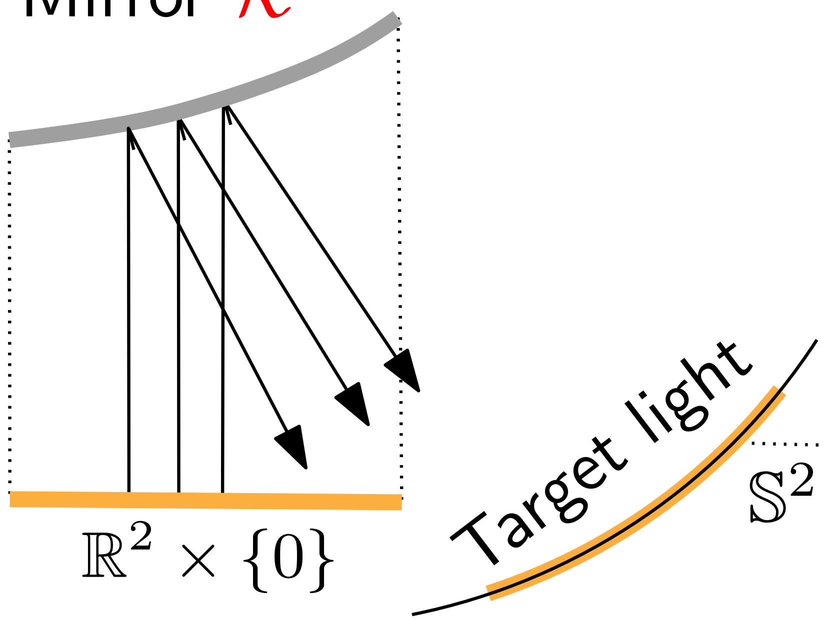
Numerical results: mirror design / collimated light



Target image: $N = 256 \times 256$ Diracs

Numerical results: mirror design / collimated light



Collimated source

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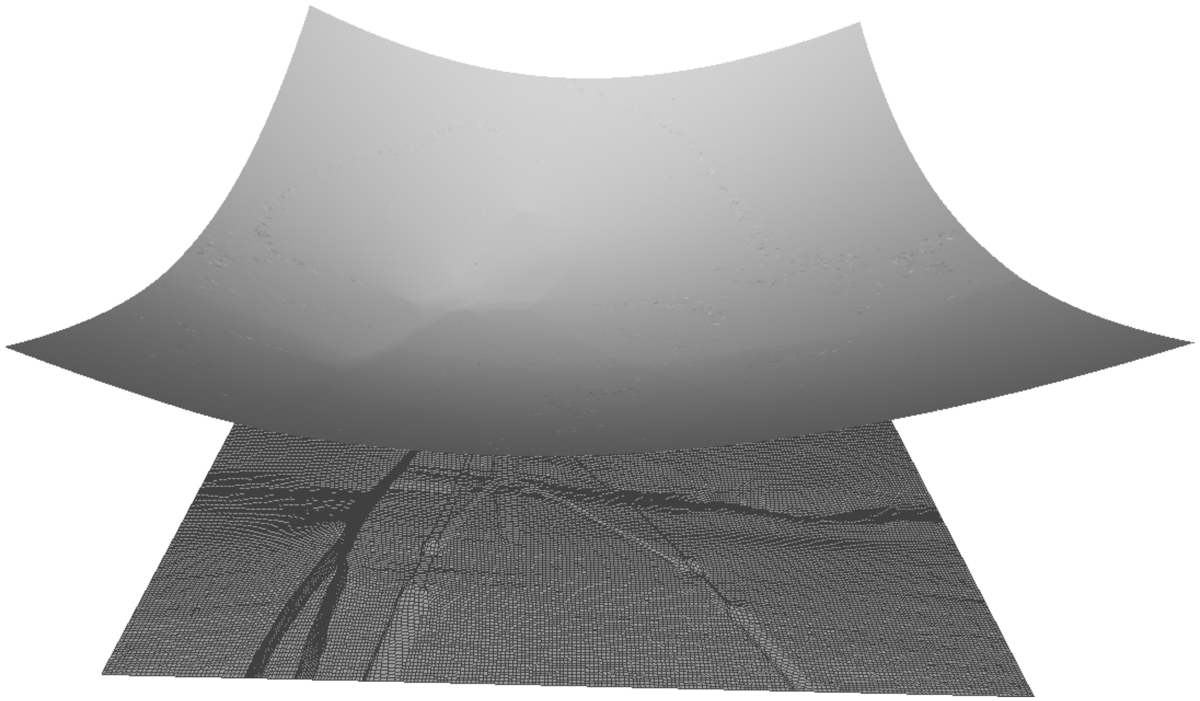
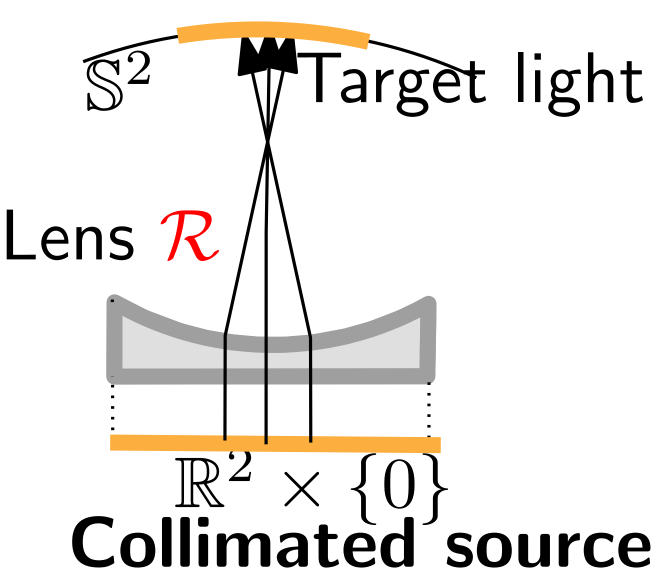
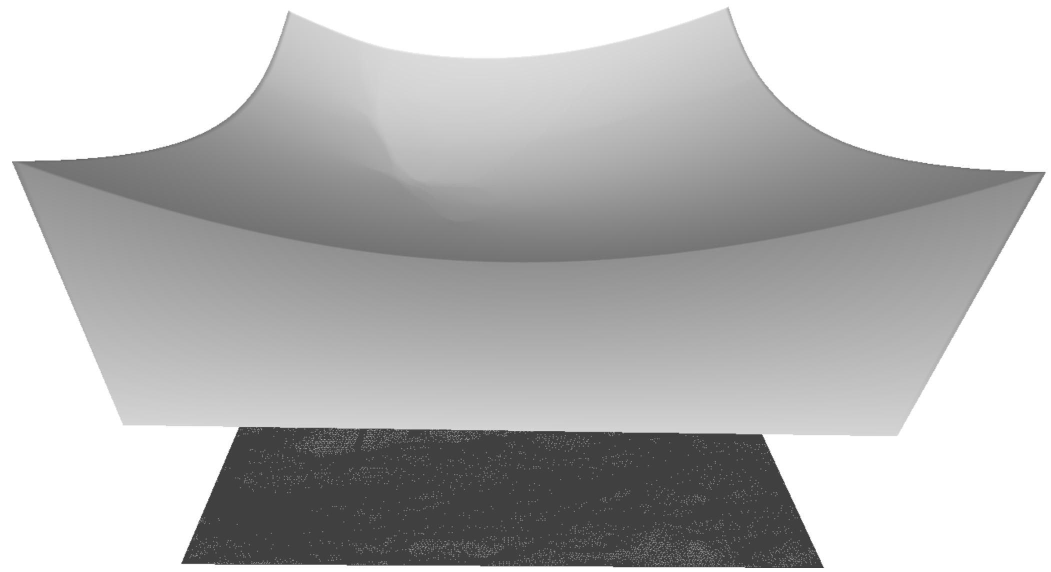


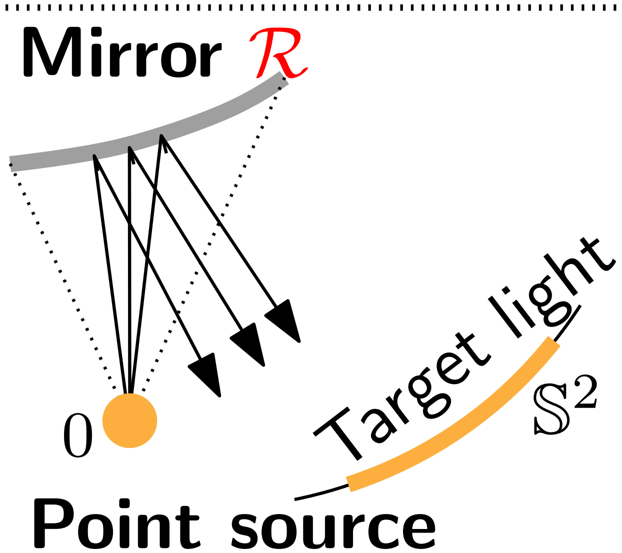
Laguerre diagram and mesh

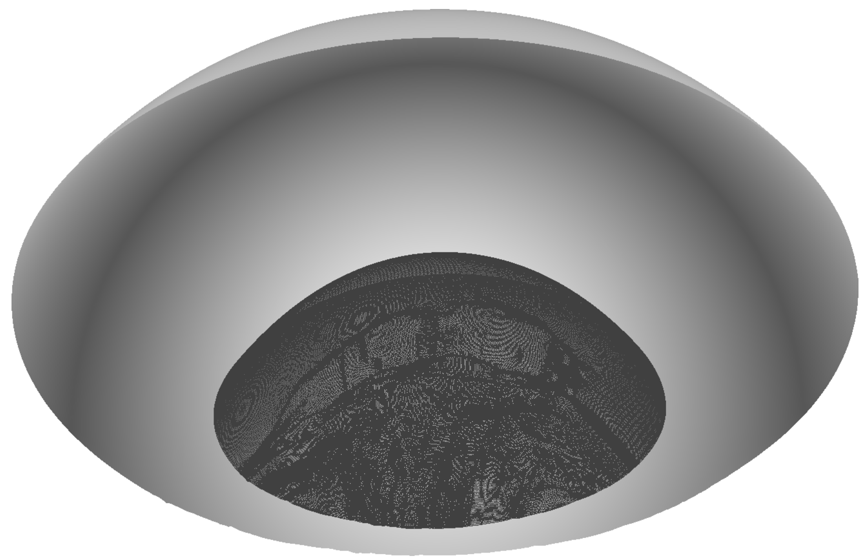
Reflected image

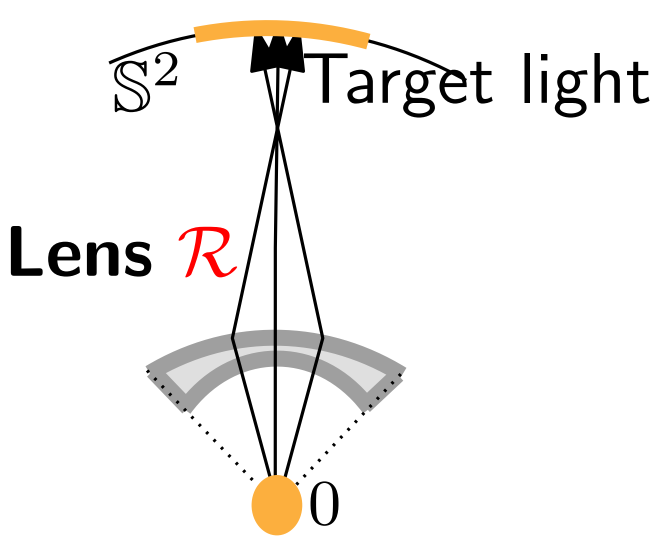
Numerical results: other settings

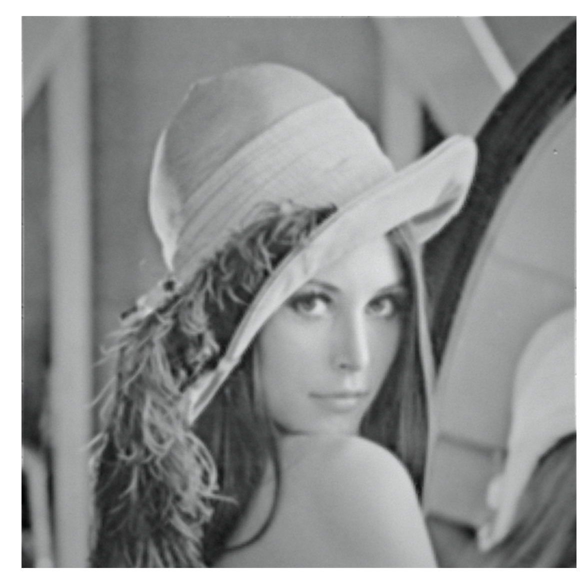












Numerical results: physical prototypes

Summary & Perspectives

- **Optimal transport** can be used to **unify** non-imaging optics problems
- **Optimal transport** can be solved **very efficiently** using Newton algorithm

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Thank you for your attention