Linear Chirp based Retrieval

Mode retrieval using time-frequency coefficients

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Structure of the presentation

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2. Continuous signals
3. Linear Chirp based Retrieval (LCR)
4. Discrete and finite length signal
5. Grid restriction problem
6. Results

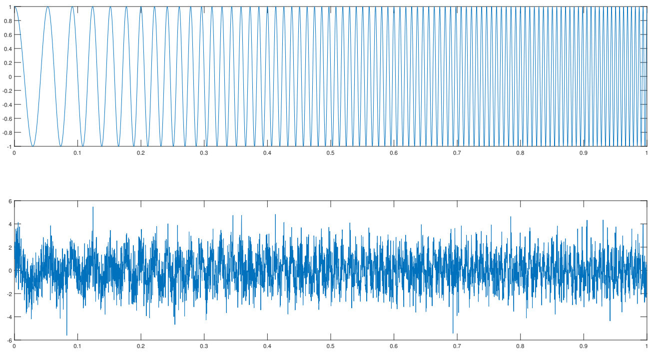
Overview

Signal example

$f(t) = \cos(2\pi(15t + 150\frac{t^2}{2}))$

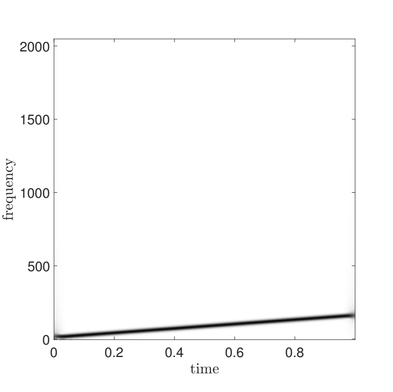
$noise(t) \sim \mathcal{N}(0, s)$

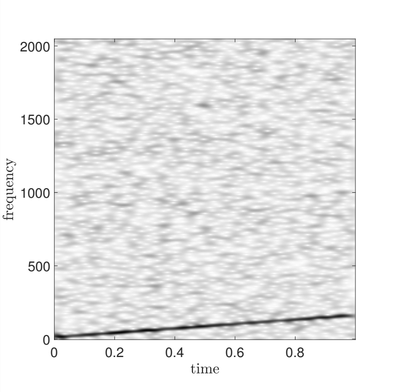
The noise is independent and identically distributed (i.i.d)



Time Frequency (TF)

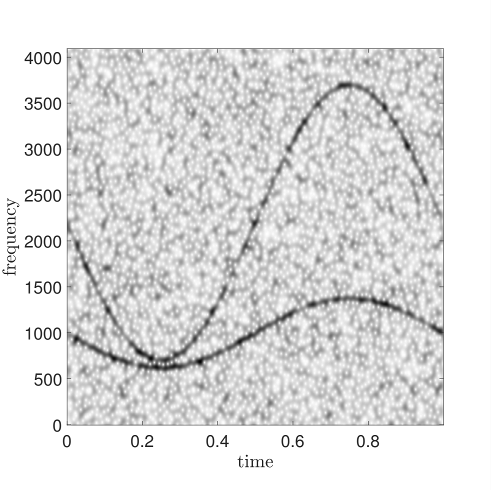
$f(t) = \cos(2\pi(15t + 150\frac{t^2}{2}))$
$noise(t) \sim \mathcal{N}(0, s)$



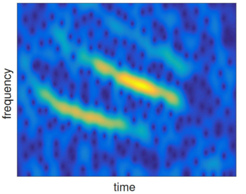


More complex signals

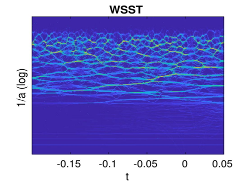
Multiple modes with oscillating frequency



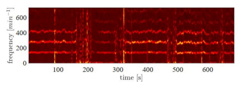
Other example of signals



Bat echolocation



Gravitational waves



Heart rate

Continuous signals

Shot Time Fourier Transform (STFT)

For $f \in L^1(\mathbb{R}), g \in L^\infty(\mathbb{R})$:

$$V_f^g(t,\eta) := \int_{\mathbb{R}} f(\tau)g(\tau - t)e^{-2i\pi\eta(\tau - t)}d\tau \tag{1}$$

Inversion formula, assuming g is real such that $g(0) \neq 0$:

$$f(t) = \frac{1}{g(0)} \int_{\mathbb{R}} V_f^g(t,\eta)d\eta. \tag{2}$$

Assuming f and $g \in L^2(\mathbb{R})$ such that $\|g\|_2 = 1$, one has the alternative reconstruction formula :

$$f(t) = \int\int_{\mathbb{R}^2} V_f^g(u,\eta)g(t-u)e^{i2\pi\eta(t-u)}dud\eta. \tag{3}$$

Multicomponent Signals (MCSs)

We considere P modes with the form $f_p(t) := A_p(t)e^{2i\pi\phi_p(t)}$

$$f(t) := \sum_{p=1}^{P} f_p(t) \tag{4}$$

With assumptions :

- instantaneous amplitude (IA) $A_p(t) > 0$ with slow variation
- instantaneous frequency (IF) $\phi_p'(t) > 0$ for all p, and such that $\phi_{p+1}'(t) - \phi_p'(t) > 0$
- the modes are separated in frequency with resolution Δ

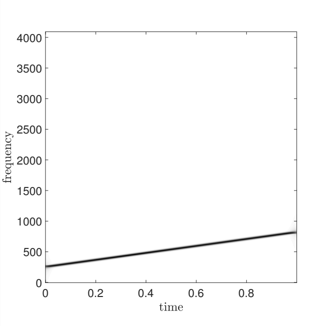
Linear Chirp based Retrieval (LCR)

Linear Chirp

Linear Chirp :

$$f(t) = Ae^{2i\pi(at+b\frac{t^2}{2})} \quad (5)$$

When $a = 250$ and $b = 568$:



To proceed with mode retrieval, we mainly rely on the information in the STFT.

Recall

$\forall r \in \mathbb{R}^+, \forall \theta s.t. \cos(\theta) \geq 0$, we consider $z = re^{i\theta}$,

$$\mathcal{F}(t \mapsto e^{-\pi z t^2})(\eta) = e^{-\frac{1}{2}} e^{-i\frac{\theta}{2}} e^{-\frac{\pi}{z}\eta^2} \tag{6}$$

The proof is the same as in the case $z \in \mathbb{R}_+^*$.

Objective : express the STFT of a linear chirp like (6)

STFT of a linear chirp 1

We recall $f(t) = Ae^{2i\pi(at+b\frac{t^2}{2})}$, therefore

$f(\tau) = Ae^{2i\pi\left(\phi(t)+(\tau-t)\phi'(t)+\frac{(\tau-t)^2}{2}\phi''(t)\right)}$

We choose $g(t) = e^{-\pi\frac{t^2}{\sigma^2}}$

$$\begin{aligned}
V_f^g(t,\eta) = &\int_{\mathbb{R}} Ae^{2i\pi\left(\phi(t)+(\tau-t)\phi'(t)+\frac{(\tau-t)^2}{2}\phi''(t)\right)} e^{-\pi\frac{(\tau-t)^2}{\sigma^2}} e^{-2i\pi\eta(\tau-t)} d\tau \\
=&Ae^{2i\pi\phi(t)} \int_{\mathbb{R}} e^{-\pi\left(\frac{1}{\sigma^2}-i\phi''(t)\right)(\tau-t)^2} e^{-2i\pi(\eta-\phi'(t))(\tau-t)} d\tau \\
=&Ae^{2i\pi\phi(t)}\mathcal{F}\left(e^{-\pi\left(\frac{1}{\sigma^2}-i\phi''(t)\right)(\tau-t)^2}\right)(\eta-\phi'(t))
\end{aligned}$$

We can apply (6) with $z = \left(\frac{1}{\sigma^2} - i\phi''(t)\right)$

STFT of a linear chirp 2

The following expression can then be deduced :

$$V_f^g(t,\eta) = V_f^g(t,\phi'(t))e^{\frac{-\pi\sigma^2(1+i\phi''(t)\sigma^2)}{1+(\phi''(t)\sigma^2)^2}(\eta-\phi'(t))^2} \tag{7}$$

This expression :

- Has a STFT independent of η
- Has two unkowns : ϕ', ϕ'' (with respect to the mode retrieval problem)

Modulation operators 1

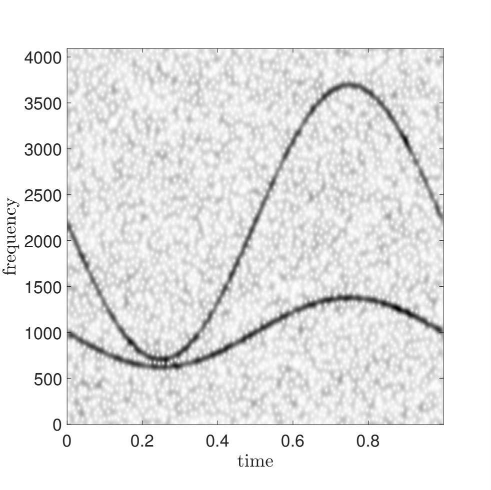
Using modulation operators :

$$\hat{\omega}_f^{[2]} : (t, \eta) \mapsto \hat{\omega}_f^{[2]}(t, \eta) = \phi'(t)$$
$$\hat{q}_f : (t, \eta) \mapsto \hat{q}_f(t, \eta) = \phi''(t)$$

Theoretically, this is true for all η, so it could be chosen arbitrarly.
In practice, η should be close to $\phi'(t)$.

Modulation operators 2

Ridge detection technique can be used to get $\varphi'(t) \approx \phi'(t)$. On the example below, one can choose the P highest values in frequency :



General MCS case

The equation of the LC based STFT (7) and modulation operators $\hat{\omega}^{[2]}$, $\hat{q}$ are approximations :

- When f is not a linear chirp
- When the signal contains noise
- When the signal contains multiple modes

Computation of the operators

Operators $\widehat{\omega}_f^{[2]}$ and $\widehat{q}_f$ can be computed with STFTs only.

With $\widehat{q}_f = \Re\{\tilde{q}_f\}$:

$$\tilde{q}_f(t,\eta) = \frac{1}{2i\pi}\frac{V_f^{g''}(t,\eta)V_f^{g}(t,\eta) - \left(V_f^{g'}(t,\eta)\right)^2}{V_f^{tg}(t,\eta)V_f^{g'}(t,\eta) - V_f^{tg'}(t,\eta)V_f^{g}(t,\eta)} \quad (8)$$

$$\widehat{\omega}_f^{[2]}(t,\eta) = \Re\left\{\eta - \frac{1}{2i\pi}\frac{V_f^{g'}(t,\eta)}{V_f^{g}(t,\eta)} + \tilde{q}_f(t,\eta)\frac{V_f^{tg}(t,\eta)}{V_f^{g}(t,\eta)}\right\} \quad (9)$$

Discrete and finite length signal

Considered signal

Using N frequency bins, for $n = 0, \cdots, L-1$:

- f is the discrete sequence such that $f[n] = f(\frac{n}{L})$
- $(g[n])_{n\in\mathbb{Z}}$ are the samples at $\frac{n}{L}$ of the Gaussian window

$(g[n])$ is further truncated to be supported on $\{-M, \cdots, M\}$ such that $2M+1 \leq N$

The objective is to retrieve all f_p modes of f.

Discrete STFT

STFT in the discrete case :

$$\mathrm{V}_{f_p}^{g}[m,k] = \sum_{n=-M}^{n=M} f_p[m+n]g[n]e^{-2i\pi\frac{kn}{N}} \tag{10}$$

Inversion formula :

$$f_p[m] = \frac{1}{g(0)}\sum_{k=0}^{N-1} \mathrm{V}_{f_p}^{g}[m,k], \tag{11}$$

Assuming f is L-periodic :

$$f_p[m] = \frac{\sum\limits_{q=m-M}^{m+M}\sum\limits_{k=0}^{N-1} \mathrm{V}_{f_p}^{g}[q \bmod L, k]g[m-q]\frac{e^{i2\pi\frac{k(m-q)}{N}}}{N}}{\sum\limits_{q=m-M}^{q=m+M} g[m-q]^2}. \tag{12}$$

Notations

To simplify notations, we set :

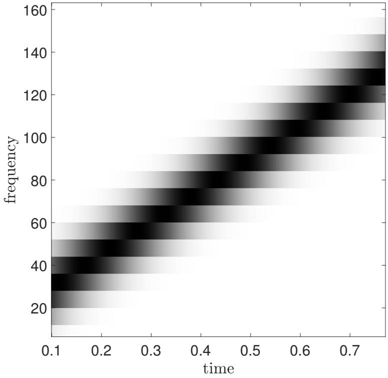
$$\psi'_p[m] := \widehat{\omega}_f^{[2]}[m, \varphi_p[m]]$$
$$\psi''_p[m] := \widehat{q}_f[m, \varphi_p[m]]$$

approximating $\phi'_p(\frac{m}{L})$ and $\phi''_p(\frac{m}{L})$ respectively.

Grid restriction problem

Mode retrieval : finite values of STFT

$\psi'_p(t)$ is not necessarly a value of the frequency bins



Deduction of the STFT 1

Setting $k_0 := \lfloor \psi_p'[m] \frac{N}{L} \rceil$
and recalling from (7) :

$$V_f^g\left(\frac{m}{L}, k_0\frac{L}{N}\right) = V_f^g\left(\frac{m}{L}, \phi'(\frac{m}{L})\right) e^{\frac{-\pi\sigma^2(1+i\phi''(t)\sigma^2)}{1+(\phi''(t)\sigma^2)^2}(k_0\frac{L}{N}-\phi'(t))^2} \tag{13}$$

we can deduce off-grid values :

$$\begin{aligned} V_f^g\left(\frac{m}{L}, \psi_p'[m]\right) \approx & V_{f_p}^g\left(\frac{m}{L}, \psi_p'[m]\right) \\ \approx & V_f^g(\frac{m}{L}, k_0\frac{L}{N})e^{\frac{\pi\sigma^2(1+i\psi_p''[m]\sigma^2)}{1+(\psi_p''[m]\sigma^2)^2}(k_0\frac{L}{N}-\psi_p'[m])^2} \\ \approx & \frac{1}{L}\mathrm{V}_f^g[m, k_0]e^{\frac{\pi\sigma^2(1+i\psi_p''[m]\sigma^2)}{1+(\psi_p''[m]\sigma^2)^2}(k_0\frac{L}{N}-\psi_p'[m])^2}. \end{aligned} \tag{14}$$

Deduction of the STFT 2

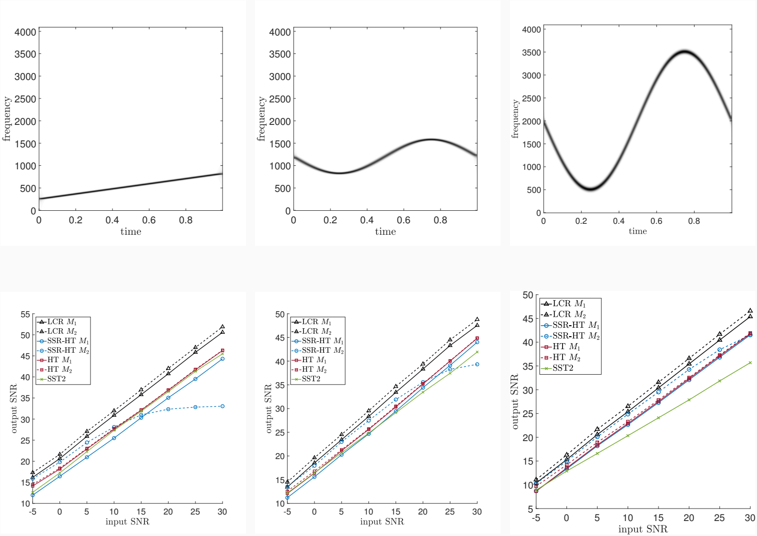
$$
\begin{aligned}
V_{f_p}^{g}[m,k] \approx & V_f^g\left(\frac{m}{L},\psi_p'[m]\right) e^{\frac{-\pi\sigma^2(1+i\psi_p''[m]\sigma^2)}{1+(\psi_p''[m]\sigma^2)^2}(k\frac{L}{N}-\psi_p'[m])^2} \\
\approx & V_f^g[m,k_0] e^{\frac{\pi\sigma^2(1+i\psi_p''[m]\sigma^2)}{1+(\psi_p''[m]\sigma^2)^2}\left[(k_0\frac{L}{N}-\psi_p'[m])^2-(k\frac{L}{N}-\psi_p'[m])^2\right]} \\
\approx & V_f^g[m,k_0] e^{\frac{\pi\sigma^2(1+i\psi_p''[m]\sigma^2)}{1+(\psi_p''[m]\sigma^2)^2}\left[\frac{L(k_0-k)}{N}(\frac{L(k_0+k)}{N}-2\psi_p'[m])\right]}.
\end{aligned}
\tag{15}
$$

Results

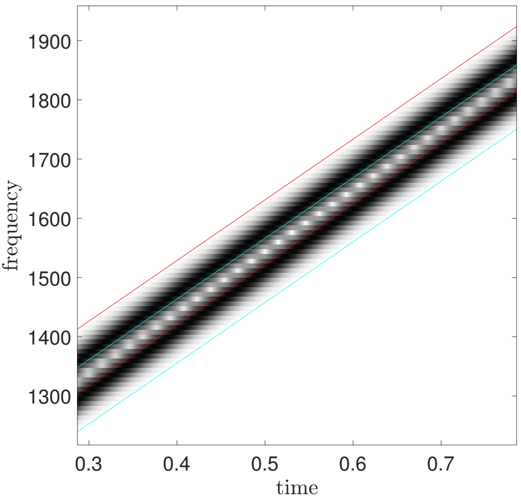
Comparison with other techniques

- Hard Thresholding (HT)
- Shifted-Symmetrized-Regularized Hard-Thresholding HT (SSR-HT)
- second-order synchrosqueezing transform (SST2)

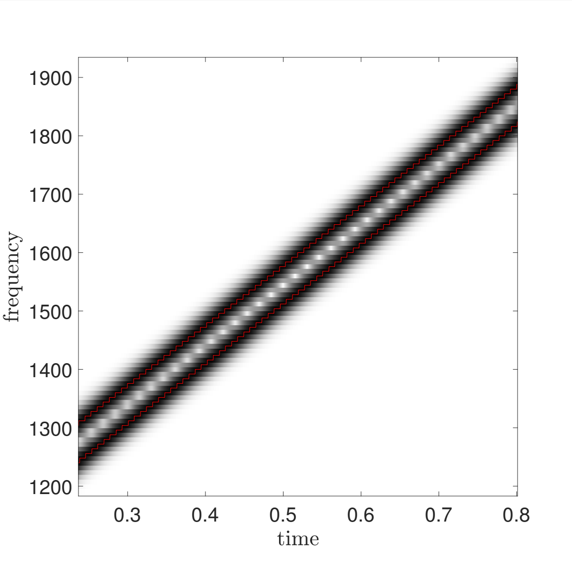
SNR on single component signals



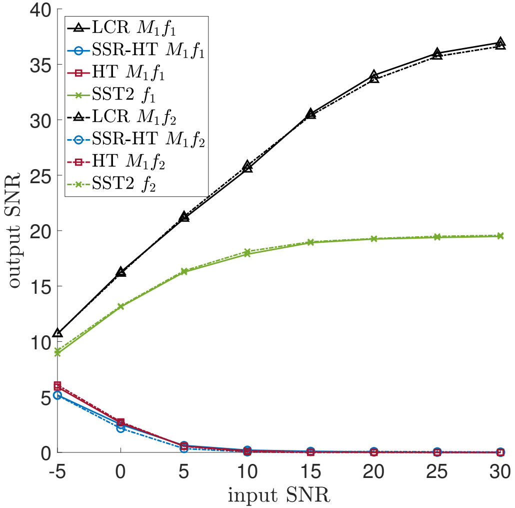
Mode mixing


1900
1800
1700
1600
1500
1400
1300
frequency
0.3
0.4
0.5
0.6
0.7
time

Avoid mode mixing



SNR on a MCS



LCR $M_1 f_1$
SSR-HT $M_1 f_1$
HT $M_1 f_1$
SST2 f_1
LCR $M_1 f_2$
SSR-HT $M_1 f_2$
HT $M_1 f_2$
SST2 f_2
output SNR
input SNR
40
35
30
25
20
15
10
5
0
-5
0
5
10
15
20
25
30